

Heuristic Projections of Solvency and Contribution Risks

Due to Non-Stationary Stochastic Rates of Return

- in view of Optimal Pension Funding-

성주호
Joo-Ho

Sung

본 연구는 확정급여형 적립방법론에서 최근 관심이 증대되고 있는 그러나 연구가 미진한 투자수익률의 불안정성 문제(non-stationary problem)을 최적통제이론(discrete optimal control theory) 관점에서 다루고 있다. 접근방법은 선행연구(Haberman & Sung (1994, 2002, 2004))을 따르고 있지만, 항시적 안전성(continued solvency)을 추구하는 성숙형 연기금(mature pension scheme)의 투자 특성을 반영하기 위하여 시계열분석에서 자주 접하게 되는 추세불안정성(trending non-stationarity)을 다루고 있다는 점, 연기금 평가모형이 계속기업의 공준에 근거한다는 점 그리고 국제회계기준(IAS) 및 유럽 등에서 가장 보편적으로 권장하고 있는 이연상각방법(spread method)의 최선적 대안을 제시하고 있다는 점에서 그 차별성을 찾을 수 있다. 아울러, Sung(2003)의 확정모형(deterministic model)을 통계모형(stochastic model)으로 확장함으로써, 이론적 · 현실적 적용 가능성을 제고하고 있음에 주목할 필요성이 있다. 물론, 선행연구에서처럼 연기금 평가상에 직면하게 되는 다음의 두 가지 대별적 위험을 주된 관리대상 위험으로 설정하여 이들 위험을 최선적으로 (본고에서는 최적이라 할 수 없으므로) 극소화하는 적립전략을 도출함에 목적을 두고 있다: 기여율의 변동성을 측정하는 기여위험(contribution (rate) risk)과 약속된 급여의 불안정성을 측정하는 지급능력위험(solvency risk). 아울러, 이들 위험간의 상호 보완성을 장기추계치(long-term projections)로 예시함으로써 통제이론의 적용 가능성 및 그 시사점을 제시하고 있다. 끝으로, 본고는 공적연금 뿐만 아니라 기업연금에서 흔히 접하는 금융시장의 불안정성 문제를 재원조달관점에서 효율적으로 접근하는 작은 출발점이 되기를 희망한다.

국문색인어: 확정급여형, 연기금 적립 및 평가, 추세 불안정성, 최적통제이론, 이연상각방법, 장기추계, 기여위험, 지급능력위험

I. Introduction

We consider the dynamic funding of defined benefit pension plans based on the spread method for eliminating surpluses and deficiencies. Haberman & Sung (1994, 2002) have demonstrated a dynamic programming formulation for determining the contribution rate with an unconstrained structure for the case of a solvency valuation (i.e. short-term, winding-up valuation).

As noted by Dufresne (1988) and Owadally & Haberman (1999), the actuarial profession, in the UK and elsewhere, has a strong interest in pension funding process which uses the spread method for amortizing the actuarial surpluses/deficits including the initial unfunded liability over a moving term and, in particular, for considering the modelling of a time-varying economic environment. Further, Owadally & Haberman (1999) demonstrate the conditions under which the use of moving term amortization is efficient. However, these conventional approaches are static rather than dynamic.

In general, modelling the time-varying rates of investment return for the assets of a pension scheme, i_{t+1} , would be based on the forecasting of its respective mathematical trend curve through the analysis of a related historical data series. Further, the respective projections would depend somewhat on the perspectives of the pension experts (especially, the actuary and investment manager) regarding the future investment market movements, for example, in terms of optimistic, expected and pessimistic scenarios. In this paper, we allow for trends in i_{t+1} with respect to time t recognising that the major source of instability in pension funding is the change in investment rates of return over time. We illustrate the methodology using a particular stochastic model.

We focus attention on how the pension funding plan using the spread method can optimally recognize all of the solvency surpluses or deficits arising from the volatility in stochastic investment returns, in the light of optimal control theory, for the case where the control time horizon is infinite and the funding plan is applied to a classical actuarial valuation (i.e. a long-term, going-concern valuation). The spread method is presented in terms of the proportional parameter k_t , which is considered to be a controlling parameter. Moreover, this controller is treated to be a stochastic controlling variable, rather than a deterministic controlling variable as in Sung (2003).

The remainder of this paper is organised as follows: In section II, we describe the model to be used. In section III, we formulate the non-stationary linear quadratic performance (NSLQP) optimisation problem but note that it is insoluble. In section IV, we propose a heuristic optimisation procedure for this non-stationary control problem and then give the heuristic solutions and some valuable comments in section V and VI, respectively. Lastly, we provide some illustrative numerical examples and some suggestive concluding remarks in section VII.

II. Model Construction and Assumptions

1. Mathematical Model

As with any model of a real world problem, it is necessary to make a number of simplifying assumptions in order that we may focus effectively on the key features of the problem to be solved. We work in discrete time, that is valuations are carried out annually so that, at time $t \in \Omega (\equiv \{0, 1, 2, \dots\})$, an actuarial valuation is conducted to evaluate the

AL_t : the actuarial liability at time t , in respect of all members at time t

F_t : the size of the scheme funds at time t , measured in terms of the market value of the underlying assets.

C_t : the contribution in year $(t, t+1)$, which we assume to be payable at time t .

B_t : the benefit outgo in year $(t, t+1)$, which we assumed to be payable at time t . We will assume that the underlying demographic assumptions are exactly realised in the projections so that this term is not a source of surplus or deficiency.

NC_t : the normal cost which would be the contribution if all the actuarial assumptions were realised exactly.

We work with a simplified pension scheme where the only benefit offered is a normal retirement pension, with a level based on the member's final salary. We assume that there is a single age at entry (a) and a single age of retirement (r). It is assumed that the number of new entrants and the level of salaries grow geometrically and that this has applied for a sufficiently long period of time: and hence, we consider a stable population.

Then, the following recurrence relations follow

$$F_{t+1} = (1+i_{t+1}) \cdot (F_t + C_t - B_t) \quad (1)$$

$$AL_{t+1} = e^{\eta} \cdot (AL_t + NC_t - B_t) \quad (2)$$

$$AL_{t+1} = e^{\alpha+\beta} \cdot AL_t \quad (3)$$

where i_{t+1} is specified below and defined in a (nominal or real) manner consistent with F_t ; η is the force of interest corresponding to the valuation interest rate i_v , assumed to be constant for all t ; α is the force of membership growth and β is the force of salary growth.

As further notation, we introduce the funding ratio $FR_t = \frac{F_t}{AL_t}$, the contribution ratio

$$CR_t = \frac{C_t}{AL_t}, \text{ and the benefit ratio } BR_t = \frac{B_t}{AL_t}.$$

2. Damped harmonic motion of investment rates of return

In order to illustrate the methodology, we consider a specific time dependent stochastic model for i_{t+1} which is non-stationary rather than the general case.

As suggested by Loades (1992), the historical trend curve of investment rates of return may be characterised by a harmonic (or periodic) curve related to a series of business/economic cycles. And so, we propose that, for a mature pension scheme with matched investments, the future trajectory of the investment rates of return will not be subject to significant fluctuations about the principal trend. This assumption indicates conceptually that the pension fund portfolio is focusing on hedging against its investment risks, over time. Hence, we propose the following specific model characterised by trending non-stationarity: for all $t \in \Omega$,

$$1+i_{t+1} = 1 + i_e + \text{sinc}_{t+1} + \varepsilon_{t+1}, \text{ in which} \quad (4)$$

i_e = expected long-term stabilised rate of return on future investment projected from the historical linear trend (from the viewpoint of classical actuarial valuations, i_e would be often used as the valuation interest rate i_v); $\text{sinc}_{t+1} = \sin(\omega \cdot t + \varpi) / (\omega \cdot t + \varpi)$, in which ω = the angular frequency = $2\pi/\phi$, $\phi(>0)$ denotes the period of the business cycle, ϖ is the initial phase shift of the business cycle and $(\omega \cdot t + \varpi)^{-1}$ is the damped amplitude of the oscillations in i_{t+1} . The sinc-function provides a simple way of modelling damped deterministic oscillations (for properties of the sinc-function, see McGillem & Cooper (1991, section 3.8)); and $\varepsilon_{t+1} \sim \text{iid } N(0, \sigma^2)$ (with $\sigma^2 < \infty$), which represents a stochastic short-term fluctuation about the long-term periodic curve sinc_{t+1} , arising from the uncertainties inherent in investment markets.

Hence, the non-stationary Gaussian process $\{i_{t+1}\}$ is represented as the sum of a stationary Gaussian process, $\{\varepsilon_{t+1}\}$, and a deterministic damped harmonic trend convergent to i_e , $\{i_e + \text{sinc}_{t+1}\}$. It is worth noting that our process specified in (4) can be thought of as a non-stationary stochastic version of the deterministic model proposed by Sung (2003).

3. Spread Controlling Variable

In order to respond successfully to the non-stationary situations of investment markets, we abandon the assumption of a constant spread controlling parameter (as in Dufresne (1988), for example) and instead allow the spread controlling parameter applying to the unit control period $[t, t+1)$, k_t , to depend on the currently available information. So, k_t can be expressed as a time-varying function of the currently available information vector, say \mathfrak{I}_t , where

$$\mathfrak{I}_t = (FR_0, FR_1, \dots, FR_t, CR_0, CR_1, \dots, CR_{t-1}); \text{ and } k_t = k_t(\mathfrak{I}_t) \text{ for all } t \quad (5)$$

Here, we employ the general and well-recognised spread funding formula characterised by

$$C_t = NC_t + k_t (AL_t - F_t), \text{ in which } 0 < k_t < 1 \text{ for all } t \in \Omega.$$

And then, dividing by AL_t and applying equation (5), we have

$$CR_t = NR_t - k_t(\mathfrak{I}_t) \cdot (FR_t - 1), \quad (6)$$

in which NR_t is previously defined as in a form of $NR_t = \frac{NC_t}{AL_t}$ (i.e. normal cost ratio) and

$$k_t(\mathfrak{I}_t) \in \{k_t(\mathfrak{I}_t): 0 < k_t(\mathfrak{I}_t) < 1\} \text{ for all } t \in \Omega.$$

Given the recurrence relations (1), (2), (3) and (6), we obtain the non-stationary dynamic system equation: for all $t \in \Omega$,

$$FR_{t+1} - 1 = n_1(t) \cdot [1 - k_t(\mathfrak{Z}_t)] \cdot (FR_t - 1) + n_2(t) \quad \text{with given } FR_0 - 1, \quad (7)$$

in which

$$k_t(\mathfrak{Z}_t) \in \{k_t(\mathfrak{Z}_t): 0 < k_t(\mathfrak{Z}_t) < 1\},$$

$$n_1(t) = (1 + i_{t+1}) / \exp(\alpha + \beta) \sim \text{iid } N(\mu_1(t), \sigma_1^2) \text{ and}$$

$$n_2(t) = (i_{t+1} - i_v) / \exp(\eta) \sim \text{iid } N(\mu_2(t), \sigma_2^2), \text{ in which}$$

$$\mu_1(t) = (1 + i_e + \text{sinc}_{t+1}) / \exp(\alpha + \beta), \quad \mu_2(t) = (i_e + \text{sinc}_{t+1} - i_v) / \exp(\eta),$$

$$\sigma_1^2 = \sigma^2 / \exp(2\alpha + 2\beta) \text{ and } \sigma_2^2 = \sigma^2 / \exp(2\eta).$$

Hence, this dynamic model is a stochastic difference equation of order one, which will sequentially generate $\{FR_{0-1}, FR_{1-1}, FR_{2-1}, FR_{3-1}, \dots\}$ with certainty as control actions $\{k_0(\mathfrak{Z}_0), k_1(\mathfrak{Z}_1), k_2(\mathfrak{Z}_2), \dots\}$ are taken. Further, $\{FR_{0-1}, FR_{1-1}, FR_{2-1}, FR_{3-1}, \dots\}$ is a discrete-time, infinite-state Markov process for the reason that $\Pr[FR_{t+1-1}, FR_{t+2-1}, FR_{t+3-1}, \dots | FR_0-1, FR_1-1, FR_2-1, \dots, FR_t-1] = \Pr[FR_{t+1-1}, FR_{t+2-1}, FR_{t+3-1}, \dots | FR_t-1]$ for all $t \in \Omega$, so that FR_t-1 is a state variable of the controlled dynamics (7). It is thus sufficient to determine $k_t(\mathfrak{Z}_t)$ as a function of FR_t-1 , i.e. $k_t(\mathfrak{Z}_t) = k_t(FR_t-1)$, so that equation (7) can be rewritten as follows: for all $t \in \Omega$,

$$FR_{t+1}-1 = n_1(t) \cdot (1 - k_t(FR_t-1)) \cdot (FR_t-1) + n_2(t) \quad \text{with given } FR_0-1 \quad (8)$$

where $k_t(FR_t-1) \in \{k_t(FR_t-1): 0 < k_t(FR_t-1) < 1\}$ and the others are the same as in (7).

III. NSLQP Optimisation Control Problem

As noted in Haberman (1994), $k_t(FR_t)$ can be interpreted as a penal rate of interest charged on the mismatch between FR_t and the target funding ratio (i.e. 100%), so that equation (6) can be thought of as being designed to spread CR_t around NR_t through the use of $k_t(FR_t)$. Applying the dynamic programming (DP) tool, we need firstly to formulate the performance index. The following index, Λ_θ , would be designed to give an undiscounted weighted penalty to the funding ratio mismatch (i.e. solvency risk) as well as the contribution ratio mismatch (i.e. contribution risk), resulting from the current control action $k_t(FR_t)$: that is,

$$\begin{aligned} \Lambda_\theta &= E \left\{ \sum_{t=0}^{\infty} [\theta \cdot (FR_{t+1} - 1)^2 + (1-\theta) \cdot (CR_t - NR_t)^2] \right\} \quad (\text{by applying (6)}) \\ &= E \left\{ \sum_{t=0}^{\infty} [\theta \cdot (FR_{t+1} - 1)^2 + (1-\theta) \cdot k_t(FR_t-1)^2 \cdot (FR_t - 1)^2] \right\} \end{aligned}$$

where $\theta \in (0, 1)$ is a convex combination parameter which is used to provide a trade off between the solvency risk and the contribution risk.

In accordance with the above discussion, we can construct the following stochastic infinite-time horizon NSLQP optimisation control problem which is different to Sung (2003): that is,

$$\text{Min}_{\{k_t(FR_t-1); t=0,1,2,\dots\}} E \left\{ \sum_{t=0}^{\infty} [\theta \cdot (FR_{t+1} - 1)^2 + (1-\theta) \cdot k_t(FR_t-1)^2 \cdot (FR_t - 1)^2] \right\} \quad (9)$$

subject to $\theta \in (0, 1)$ and controlled object governed by the system equation (8)

In general, there are limits to the usefulness of the DP methodology. The stronger are the constraints on the controlling variables (e.g. $\{k_t(FR_t-1): 0 < k_t(FR_t-1) < 1, \forall t\}$), the longer is the control horizon (e.g. Ω) and the higher is the level of non-stationarity present (e.g. time-varying $n_1(t)$ and $n_2(t)$, $\forall t$), then the higher is the potential risk that the corresponding optimisation problem

is insoluble. Thus, it is straightforward to demonstrate that the DP tool can not provide a solution to the above problem (9) because the infinite optimisation procedure with respect to $k_0(FR_0-1)$, $k_1(FR_1-1)$, $k_2(FR_2-1)$, ... requires an infinite number of optimising computations because of the strong constraint that $0 < k_t(FR_t-1) < 1$, $\forall t$, and because of the presence of non-stationarity.

IV. Heuristic Optimisation Approach

In order to obtain a physically admissible optimisation procedure by applying DP over an infinite-time control horizon, we need to weaken both the constraint and the non-stationarity. As a heuristic optimisation approach to the above control problem (9), we firstly reformulate the space $\{k_t(FR_t-1): 0 < k_t(FR_t-1) < 1, \forall t\}$ into the following form: for some large $\varepsilon > 0$,

$$\{k_t(FR_t-1): |k_t(FR_t-1)| \leq \varepsilon\} \quad (10)$$

(the redefined space leads to a suggested interpretation of the result as mentioned in section 4.4).

Secondly, we reformulate the non-stationary controlled object (7) into the controlled threshold object (14) as defined in subsection 4.1.

1. Threshold Model

Given the above comment on the insolubility of the control problem (9), we propose a reasonable approximate model for the non-stationary controlled object, which can be solved computationally.

As an approximation to the system equation (7), we propose a threshold system equation, which involves a switching rule from the non-stationary stochastic case to the stationary stochastic case at some pre-specified point of time. Accordingly, the controlled object governed by this threshold system equation will be called the controlled threshold object; here, the term 'threshold' is adopted from 'threshold autoregressive models' in the area of time series analysis (for details and examples, see Tong & Lim (1980)). Its mathematical specification takes the steps described below. Regarding notation, we use the upper prefix T to denote the threshold LQP optimisation problem.

- (Step 1: Analysis of the time series)

: The non-stationary property of equation (7) is completely determined by the stochastic process $\{i_{t+1}\}$ governed by model (4). And further, its expectation $\{E(i_{t+1})\}$ is characterised as being damped and ultimately convergent to its limiting value i_e . Of course, the value of t required for this limit value to be attained will not be finite, and so we need to find, from an approximation point of view, a point of time (denoted by t^* , which shall be called the threshold time value) after which the main trends remain "relatively unchanged". In this respect, we can set an approximation criterion for determining t^* .

- (Step 2: Approximation trend criterion for t^*)

: For a chosen approximation trend error value, denoted by $\Delta > 0$, we find t^* satisfying $|E(i_{t+1} - i_{t^*+1})| < \Delta$ for any $t = t^*+1, t^*+2, t^*+3, \dots$. In general, Δ would be required to be as small as possible to justify the appropriateness of the approximation. Our infinite control horizon can then be classified into the two disjoint and exclusive control regimes, that is, $\Omega = \Omega_1 \cup \Omega_2$ and $\Omega_1 \cap \Omega_2 = \emptyset$, where $\Omega_1 = \{0, 1, \dots, t^*-1\}$ and $\Omega_2 = \{t^*, t^*+1, t^*+2, \dots\}$;

- (Step 3: Decomposition)

: We decompose the model (4) into two different models for Ω_1 and Ω_2 , switching from the non-stationary process to the stationary process when time t reaches the threshold time value t^* . That is,

$$(1 + \dot{i}_{t+1}) = \begin{cases} 1 + i_e + \text{sinc}_{t+1} + \varepsilon_{t+1}, & \text{for each } t \in \Omega_1 \\ 1 + i_e + \varepsilon_{t+1}, & \text{for each } t \in \Omega_2 \end{cases} \quad (11)$$

Therefore, it would be consistent with the above approximation model (11) to reformulate the spread parameter function (5) in the following way. Denoting the indicator function of set S by l_S , then for each $t \in \Omega$,

$$k_t = k_t({}^TFR_{t-1}) \cdot 1_{\Omega_1} + k({}^TFR_{t-1}) \cdot 1_{\Omega_2} \quad (12)$$

We consider $k_t({}^TFR_{t-1}) \cdot 1_{\Omega_1} + k({}^TFR_{t-1}) \cdot 1_{\Omega_2}$ to be the threshold spread parameter, which is composed of two factors, that is, $k_t({}^TFR_{t-1}) \cdot 1_{\Omega_1}$ indicates the non-stationary spread parameter over Ω_1 , whereas $k({}^TFR_{t-1}) \cdot 1_{\Omega_2}$ indicates the stationary spread parameter over Ω_2 (which is allowed to vary according to the valuation information up to t , i.e. independent explicitly of time t but not constant for all t).

Hence, adopting the model represented by (11) and (12), we derive the respective threshold versions (13) and (14) of the non-stationary formula (6) and (7), respectively: that is, for each $t \in \Omega$,

$${}^T CR_t = NR_t - [k_t({}^TFR_{t-1}) \cdot 1_{\Omega_1} + k({}^TFR_{t-1}) \cdot 1_{\Omega_2}] \cdot [{}^TFR_t - 1]. \quad (13)$$

And then, the threshold system equation, which governs the controlled threshold object, is for each $t \in \Omega$,

$${}^TFR_{t+1} - 1 = [n_1(t) \cdot (1 - k_t({}^TFR_{t-1})) \cdot 1_{\Omega_1} + s_1(t) \cdot (1 - k({}^TFR_{t-1})) \cdot 1_{\Omega_2}] \cdot [{}^TFR_t - 1] + [n_2(t) \cdot 1_{\Omega_1} + s_2(t) \cdot 1_{\Omega_2}] \quad \text{with initially given } {}^TFR_0 - 1 \quad (14)$$

where,

$$|k_t({}^TFR_{t-1}) \cdot 1_{\Omega_1} + k({}^TFR_{t-1}) \cdot 1_{\Omega_2}| \leq {}^T \varepsilon,$$

both $n_1(t)$ and $n_2(t)$ are early defined in (7) and,

$$s_1(t) = (1 + i_{t+1}) / \exp(\alpha + \beta) \sim \text{iid } N(\mu_1, \sigma_1^2) \text{ and}$$

$$s_2(t) = (i_{t+1} - i_v) / \exp(\eta) \sim \text{iid } N(\mu_2, \sigma_2^2), \text{ in which}$$

$$\mu_1 = (1 + i_e) / \exp(\alpha + \beta), \mu_2 = (i_e - i_v) / \exp(\eta), \sigma_1^2 = \sigma^2 / \exp(2\alpha + 2\beta) \text{ and } \sigma_2^2 = \sigma^2 / \exp(2\eta).$$

Hence, this controlled threshold object with the parameter set $\{n_1(t), s_1(t), n_2(t), s_2(t), k_t({}^TFR_{t-1}), k({}^TFR_{t-1})\}$ has a structural change from the non-stationary controlled object, characterised by the parameter set $\{n_1(t), n_2(t), k_t({}^TFR_{t-1})\}$, to the stationary controlled object, characterised by the parameter set $\{s_1(t), s_2(t), k({}^TFR_{t-1})\}$, when the threshold time value t^* is reached.

2. Threshold LQP Optimisation Control Problem

In a similar manner to the non-stationary case formulated in section 3 above, we need to ensure that the threshold performance index (denoted by ${}^T \Lambda_\theta$) defined by the equation

$$\begin{aligned} {}^T \Lambda_\theta &= E \left\{ \sum_{t=0}^{\infty} [\theta \cdot ({}^TFR_{t+1} - 1)^2 + (1 - \theta) \cdot ({}^T CR_t - NR_t)^2] \right\} \quad \text{(by applying (13))} \\ &= E \left\{ \sum_{t=0}^{\infty} [\theta \cdot ({}^TFR_{t+1} - 1)^2 + (1 - \theta) \cdot (k_t({}^TFR_{t-1}) \cdot 1_{\Omega_1} + k({}^TFR_{t-1}) \cdot 1_{\Omega_2})^2 \cdot ({}^TFR_t - 1)^2] \right\} \end{aligned}$$

is finite.

Finally, we can set up the following approximate version of the NSLQP optimisation control problem (9). Here, this shall be called the threshold NSLQP optimisation control problem:

$$\begin{aligned} \text{Min}_{\{k_t(^TFR_t-1); t \in \Omega_1\}} E \{ \sum_{t=0}^{t^*-1} [\theta \cdot (^TFR_{t+1} - 1)^2 + (1-\theta) \cdot k_t(^TFR_t-1)^2 \cdot (^TFR_t - 1)^2] + \\ \text{Min}_{\{k(^TFR_t-1); t \in \Omega_2\}} E \{ \sum_{t=t^*}^{\infty} [\theta \cdot (^TFR_{t+1} - 1)^2 + (1-\theta) \cdot k(^TFR_t-1)^2 \cdot (^TFR_t - 1)^2] \} \} \end{aligned} \quad (15)$$

subject to given $\theta \in (0, 1)$ and controlled object governed by the threshold system equation (14)

The above control problem is characterised by a break in the control structure when switching from Ω_1 to Ω_2 . The optimisation procedure will then be considered separately over these two regimes.

3. Bellman Equations

As noted earlier, the computational insolubility of the NSLQP optimisation problem (9) comes from the infinite number of optimal functional equations and the non-stationarity. However, the threshold control problem (15) can be decomposed into two distinct problems – a non-stationary control problem over Ω_1 and a stationary control problem over Ω_2 . And so, the optimal control sequence, say $\{^*k_0(^TFR_0-1), ^*k_1(^TFR_1-1), \dots, ^*k_{t^*-1}(^TFR_{t^*-1}-1), ^*k(^TFR_{t^*}-1), ^*k(^TFR_{t^*+1}-1), \dots\}$, will be produced by solving a finite number of t^*+1 non-stationary optimal functional equations using the backward dynamic programming (BDP) approach, as illustrated below in subsection 3.1, and by solving an infinite number of stationary optimal functional equations using the forward dynamic programming (FDP) approach, as illustrated below in subsection 3.2. These approaches are each based on Bellman's principle of optimality (see, Bertsekas (1976)).

3.1 Bellman Equation in Ω_1

To derive the backward recursive equations in time $t \in \Omega_1$, we can then define

$$\begin{aligned} V(^TFR_t-1, t) \\ \equiv \text{Min}_{\{k_s(^TFR_s-1); s=t, t+1, \dots, t^*-1\}} E \{ \sum_{s=t}^{t^*-1} [\theta \cdot (^TFR_{s+1} - 1)^2 + (1-\theta) \cdot k_s(^TFR_s-1)^2 \cdot (^TFR_s - 1)^2] + \\ \text{Min}_{\{k(^TFR_s-1); s=t^*, t^*+1, t^*+2, \dots\}} E \{ \sum_{s=t^*}^{\infty} [\theta \cdot (^TFR_{s+1} - 1)^2 + (1-\theta) \cdot k(^TFR_s-1)^2 \cdot (^TFR_s - 1)^2] \} | ^TFR_t-1 \}, \end{aligned}$$

which represents the minimal expected future mismatching risk at time t , given the complete information up to time t (i.e. state variable $^TFR_t-1$).

Then, using the Bellman's principle of optimality for sequential control optimisation, we can establish the following Bellman equation: for each $t \in \Omega_1$,

$$\begin{aligned} V(^TFR_t-1, t) \\ = \text{Min}_{k_t(^TFR_t-1)} E \{ [\theta \cdot (^TFR_{t+1} - 1)^2 + (1-\theta) \cdot k_t(^TFR_t-1)^2 \cdot (^TFR_t - 1)^2] + \\ \text{Min}_{k_{t+1}(^TFR_{t+1}-1)} E \{ [\theta \cdot (^TFR_{t+2} - 1)^2 + (1-\theta) \cdot k_{t+1}(^TFR_{t+1}-1)^2 \cdot (^TFR_{t+1} - 1)^2] + \dots + \end{aligned}$$

$$\begin{aligned} & \text{Min}_{k_{t^*-1}(\text{}^T\text{FR}_{t^*-1})} E\{[\theta \cdot (\text{}^T\text{FR}_{t^*-1})^2 + (1-\theta) \cdot k_{t^*-1}(\text{}^T\text{FR}_{t^*-1})^2 \cdot (\text{}^T\text{FR}_{t^*-1})^2] + \dots | \text{}^T\text{FR}_{t+1-1}\} | \text{}^T\text{FR}_{t-1}\} \\ & = \text{Min}_{k_t(\text{}^T\text{FR}_{t-1})} E\{[\theta \cdot (\text{}^T\text{FR}_{t+1-1})^2 + (1-\theta) \cdot k_t(\text{}^T\text{FR}_{t-1})^2 \cdot (\text{}^T\text{FR}_{t-1})^2] + V(\text{}^T\text{FR}_{t+1-1}, t+1) | \text{}^T\text{FR}_{t-1}\}, \end{aligned}$$

which represents the time-varying group of t^* equations and expresses, in particular, the switching structure from the time-varying to time-invariant at time $t = t^* - 1$, i.e.

$$\begin{aligned} & V(\text{}^T\text{FR}_{t^*-1-1}, t^*-1) \\ & (16) \\ & = \text{Min}_{k_{t^*-1}(\text{}^T\text{FR}_{t^*-1})} E\{[\theta \cdot (\text{}^T\text{FR}_{t^*-1})^2 + (1-\theta) \cdot k_{t^*-1}(\text{}^T\text{FR}_{t^*-1})^2 \cdot (\text{}^T\text{FR}_{t^*-1})^2] + V(\text{}^T\text{FR}_{t^*-1}) | \text{}^T\text{FR}_{t^*-1}\}. \end{aligned}$$

3.2 Bellman Equation in Ω_2

And similarly, for each $t \in \Omega_2$,

$$\begin{aligned} & V(\text{}^T\text{FR}_{t-1}) \\ & = \text{Min}_{k(\text{}^T\text{FR}_{t-1})} E\{[\theta \cdot (\text{}^T\text{FR}_{t+1-1})^2 + (1-\theta) \cdot k(\text{}^T\text{FR}_{t-1})^2 \cdot (\text{}^T\text{FR}_{t-1})^2] + V(\text{}^T\text{FR}_{t+1-1}) | \text{}^T\text{FR}_{t-1}\}, \quad (17) \end{aligned}$$

which presents explicitly an infinite number of time-invariant optimal functional equations. However, because of the time-invariability of function $V(\cdot)$, it is sufficient to solve the optimal functional equation at time $t = t^*$ in order to specify the optimal stationary policy $k^*(\cdot)$.

Here, the Bellman equations (16) and (17) are called the threshold optimal functional equations.

V. Heuristic Solutions

The procedure solving the threshold optimal functional equation can be summarised as follows:

- the time-invariant part (17) is soluble by the FDP approach, and accordingly, we can obtain $V(\text{}^T\text{FR}_{t^*-1})$ as given in equation (18) below; and then
- by using $V(\text{}^T\text{FR}_{t^*-1})$ as a terminal condition for the time-varying part (16), it is soluble by the backward dynamic programming (BDP) approach, as given in equation (24) below (see Haberman & Sung (2004)).

Thus, we note that the optimal function value at the threshold time value t^* , i.e. $V(\text{}^T\text{FR}_{t^*-1})$, provides the conversion from the framework of the infinite-horizon threshold control problem to that of the finite-horizon threshold control problem.

Now, we shall clarify mathematically the above points (a) and (b) in turn: Firstly, the solution of the stationary part (17) is derived by the FDP approach (extending, step by step, the multistage optimising procedure to an infinite optimising procedure by applying a mathematical induction argument and then by utilising the monotone convergence theorem (see, Bertsekas (1976; section 7.1)). Then, for each $t \in \Omega_2$, we can try the solution of the Bellman equation (17) in a following quadratic form, since $\text{}^T\text{FR}_{t+1-1}$ given $\text{}^T\text{FR}_{t-1}$ is distributed with mean, $(1-k(\text{}^T\text{FR}_{t-1})) \cdot (\text{}^T\text{FR}_{t-1}) \cdot \mu_1 + \mu_2$, and variance, $[(1-k(\text{}^T\text{FR}_{t-1})) \cdot (\text{}^T\text{FR}_{t-1})]^2 \cdot \sigma_1^2 + \sigma_2^2 + 2(1-k(\text{}^T\text{FR}_{t-1})) \cdot (\text{}^T\text{FR}_{t-1}) \cdot \sigma^2 / \exp(\alpha + \beta + \eta)$, from (14). Thus, we try the following solution:

for all $t \in \Omega_2$, $V(\text{}^T\text{FR}_{t-1}) = Q_1 \cdot (\text{}^T\text{FR}_{t-1})^2 + Q_2 \cdot (\text{}^T\text{FR}_{t-1}) + Q_3$.

Hence, we can rewrite the Bellman equation (17) in $t \in \Omega_2$ as

$$\begin{aligned}
& V({}^TFR_{t-1}) \\
& = \underset{k({}^TFR_{t-1})}{\text{Min}} E \{ [\theta \cdot ({}^TFR_{t+1} - 1)^2 + (1-\theta) \cdot k({}^TFR_{t-1})^2 \cdot ({}^TFR_{t-1})^2] + Q_1 \cdot ({}^TFR_{t+1} - 1)^2 + \\
& \quad Q_2 \cdot ({}^TFR_{t+1} - 1) + Q_3 \cdot {}^TFR_{t-1} \} \\
& = \underset{k({}^TFR_{t-1})}{\text{Min}} \{ A_1({}^TFR_{t-1}) \cdot k({}^TFR_{t-1})^2 + A_2({}^TFR_{t-1}) \cdot k({}^TFR_{t-1}) + A_3({}^TFR_{t-1}) \} \quad (18) \\
& = Q_1 \cdot ({}^TFR_{t-1})^2 + Q_2 \cdot ({}^TFR_{t-1}) + Q_3, \text{ in which} \\
& A_1({}^TFR_{t-1}) = [(1-\theta) + (\theta+Q_1) \cdot (\sigma_1^2 + \mu_1^2)] \cdot ({}^TFR_{t-1})^2 \\
& A_2({}^TFR_{t-1}) = -2(\theta+Q_1) \cdot (\sigma_1^2 + \mu_1^2) \cdot ({}^TFR_{t-1})^2 - \\
& \quad [2(\theta+Q_1) \cdot (\mu_1 \cdot \mu_2 + \sigma^2 / \exp(\alpha + \beta + \eta)) + \mu_1 \cdot Q_2] \cdot ({}^TFR_{t-1}) \\
& A_3({}^TFR_{t-1}) = (\theta+Q_1) \cdot (\sigma_1^2 + \mu_1^2) \cdot ({}^TFR_{t-1})^2 + \\
& \quad [2(\theta+Q_1) \cdot (\mu_1 \cdot \mu_2 + \sigma^2 / \exp(\alpha + \beta + \eta)) + \mu_1 \cdot Q_2] \cdot ({}^TFR_{t-1}) + \\
& \quad (\theta+Q_1) \cdot (\sigma_2^2 + \mu_2^2) + \mu_2 \cdot Q_2 + Q_3
\end{aligned}$$

In order to have a unique sequence of optimal stationary controllers, $\{ {}^*k({}^TFR_{t-1}) : t \in \Omega_2 \}$, it is sufficient that for all $t \in \Omega_2$, $A_1({}^TFR_{t-1}) > 0$.

Differentiating the strictly convex function under the above condition for uniqueness, we then obtain the optimal controller for each $t \in \Omega_2$, ${}^*k({}^TFR_{t-1})$, which is written as follows:

$${}^*k({}^TFR_{t-1}) = \frac{M_1}{M_3} + \frac{M_2}{M_3} \times ({}^TFR_{t-1})^{-1}, \quad \text{in which}$$

(19)

$$M_1 = (\theta + Q_1) \cdot (\sigma_1^2 + \mu_1^2),$$

$$M_2 = \mu_1 \cdot Q_2 / 2 + (\theta + Q_1) \cdot (\mu_1 \cdot \mu_2 + \sigma^2 / \exp(\alpha + \beta + \eta)) \quad \text{and}$$

$$M_3 = (1 - \theta) + M_1.$$

We note here that the optimal stationary policy ${}^*k(\cdot)$ is not well defined when the funding ratio is 100%. In this case, without loss of generality, we adopt the rule of setting ${}^*k(\cdot)$ equal to zero in order to ensure the uniqueness of $\{ {}^*k({}^TFR_{t-1}) ; t \in \Omega_2 \}$. We note also that ${}^*k(\cdot)$ depends only on Q_1 and Q_2 . So, we need to specify Q_1 and Q_2 , which satisfy the following equations obtained by substituting (19) into (18): that is,

$$Q_1 = \frac{(1 - \theta) \cdot (\theta + Q_1) \cdot (\sigma_1^2 + \mu_1^2)}{(1 - \theta) + (\theta + Q_1) \cdot (\sigma_1^2 + \mu_1^2)}$$

(20)

$$Q_2 = \frac{(1 - \theta) \cdot [\mu_1 \cdot Q_2 + 2(\theta + Q_1) \cdot (\mu_1 \cdot \mu_2 + \sigma^2 / e^{\alpha + \beta + \eta})]}{(1 - \theta) + (\theta + Q_1) \cdot (\sigma_1^2 + \mu_1^2)}.$$

In the light of optimal control theory, equation (20) is specifically called the algebraic (equilibrium) Riccati equation, i.e. $Q_1 = f(Q_1)$ where $f(\cdot)$ is a first-degree rational function. Accordingly, the equation provides a unique and positive solution Q_1^* as given in the form (see, Whittle (1982; sections 5.3 & 5.4):

$$Q_1^* = \frac{-[1 - \theta + (2\theta - 1) \cdot (\mu_1^2 + \sigma_1^2)] + \sqrt{[1 - \theta + (2\theta - 1) \cdot (\mu_1^2 + \sigma_1^2)]^2 + 4\theta \cdot (1 - \theta) \cdot (\mu_1^2 + \sigma_1^2)^2}}{2(\mu_1^2 + \sigma_1^2)},$$

and so, the solution of Q_2 , Q_2^* , is given as follows

$$Q_2^* = \frac{2(1-\theta) \cdot (\theta + Q_1^*) \cdot (\mu_1 \mu_2 + \sigma^2 / e^{\alpha+\beta+\eta})}{(1-\theta) \cdot (1-\mu_1) + (\theta + Q_1^*) \cdot (\sigma_1^2 + \mu_1^2)} \quad (21)$$

Hence, the above uniqueness condition such that for all $t \in \Omega_2$, $A_1(\text{}^T\text{FR}_t-1) > 0$, is clearly satisfied due to $Q_1^* > 0$.

Next, we consider the non-stationary Bellman equation (16) in $t \in \Omega_1$. We apply the BDP approach and then try the solution in a quadratic form for the reason that, for each $t \in \Omega_1$, $\text{}^T\text{FR}_{t+1}-1$, given $\text{}^T\text{FR}_t-1$, is distributed with mean, $(1-k_t(\text{}^T\text{FR}_t-1)) \cdot (\text{}^T\text{FR}_t-1) \cdot \mu_1(t) + \mu_2(t)$, and variance, $[(1-k_t(\text{}^T\text{FR}_t-1)) \cdot (\text{}^T\text{FR}_t-1)]^2 \cdot \sigma_1^2 + \sigma_2^2 + 2(1-k_t(\text{}^T\text{FR}_t-1)) \cdot (\text{}^T\text{FR}_t-1) \cdot \sigma^2 / \exp(\alpha+\beta+\eta)$, from (14). Thus, $V(\text{}^T\text{FR}_t-1, t) = Q_1(t) \cdot (\text{}^T\text{FR}_t-1)^2 + Q_2(t) \cdot (\text{}^T\text{FR}_t-1) + Q_3(t)$, in which the terminal cost at time t^*-1 is given in (16) (i.e. $V(\text{}^T\text{FR}_{t^*-1}, t^*-1)$), and also the boundary conditions for BDP approach are derived as follows: that is,

$$\begin{aligned} & V(\text{}^T\text{FR}_{t^*-1}, t^*-1) \\ &= \text{Min}_{k_{t^*-1}(\text{}^T\text{FR}_{t^*-1})} E\{[\theta \cdot (\text{}^T\text{FR}_{t^*-1})^2 + (1-\theta) \cdot k_{t^*-1}(\text{}^T\text{FR}_{t^*-1})^2 \cdot (\text{}^T\text{FR}_{t^*-1})^2] + V(\text{}^T\text{FR}_{t^*-1}) | \text{}^T\text{FR}_{t^*-1}\} \\ &= Q_1(t^*-1) \cdot (\text{}^T\text{FR}_{t^*-1})^2 + Q_2(t^*-1) \cdot (\text{}^T\text{FR}_{t^*-1}) + Q_3(t^*-1), \quad \text{in which} \end{aligned} \quad (22)$$

the optimal controller at the terminal time t^*-1 is specified by

$$k_{t^*-1}(\text{}^T\text{FR}_{t^*-1}) = \frac{M_1(t^*-1)}{M_3(t^*-1)} + \frac{M_2(t^*-1)}{M_3(t^*-1)} \times (\text{}^T\text{FR}_{t^*-1})^{-1}, \quad \text{in which} \quad (23)$$

$$\begin{aligned} M_1(t^*-1) &= (\theta + Q_1^*) \cdot (\sigma_1^2 + \mu_1(t^*-1)^2), \\ M_2(t^*-1) &= \mu_1(t^*-1) \cdot Q_2^* / 2 + (\theta + Q_1^*) \cdot (\mu_1(t^*-1) \cdot \mu_2(t^*-1) + \sigma^2 / \exp(\alpha+\beta+\eta)) \quad \text{and} \\ M_3(t^*-1) &= (1-\theta) + M_1(t^*-1), \quad \text{and the boundary conditions are given in the following form} \end{aligned}$$

$$Q_1(t^*-1) = [(1-\theta) \cdot (\theta + Q_1^*) \cdot (\sigma_1^2 + \mu_1(t^*-1)^2)] / [(1-\theta) + (\theta + Q_1^*) \cdot (\sigma_1^2 + \mu_1(t^*-1)^2)] \quad \text{and} \quad (24)$$

$$Q_2(t^*-1) = (1-\theta) \cdot [\mu_1(t^*-1) \cdot Q_2^* + 2(\theta + Q_1^*) \cdot (\mu_1(t^*-1) \cdot \mu_2(t^*-1) + \sigma^2 / \exp(\alpha+\beta+\eta))] / [(1-\theta) + (\theta + Q_1^*) \cdot (\sigma_1^2 + \mu_1(t^*-1)^2)].$$

(note that it is not necessary to specify $Q_3(t^*-1)$ due to its redundancy, which will be clear from the discussion below).

In a similar manner to the stationary case, the optimal non-stationary control policy $\{k_t(\cdot) : t \in \Omega_1 - \{t^*-1\}\}$ can then be found by solving the time-varying group of t^*-1 equations by back-tracking step by step, starting with $V(\text{}^T\text{FR}_{t^*-1}, t^*-1)$ given in (22) (see Haberman & Sung (1994, 2002): that is, for each $t \in \Omega_1 - \{t^*-1\}$,

$$\begin{aligned} & V(\text{}^T\text{FR}_t-1, t) \\ &= \text{Min}_{k_t(\text{}^T\text{FR}_t-1)} E\{[\theta \cdot (\text{}^T\text{FR}_{t+1}-1)^2 + (1-\theta) \cdot k_t(\text{}^T\text{FR}_t-1)^2 \cdot (\text{}^T\text{FR}_t-1)^2] + V(\text{}^T\text{FR}_{t+1}-1, t+1) | \text{}^T\text{FR}_t-1\} \\ &= \text{Min}_{k_t(\text{}^T\text{FR}_t-1)} \{B_1(\text{}^T\text{FR}_t-1, t) \cdot (\text{}^T\text{FR}_t-1)^2 + B_2(\text{}^T\text{FR}_t-1, t) \cdot k_t(\text{}^T\text{FR}_t-1) + B_3(\text{}^T\text{FR}_t-1, t)\}, \\ &= Q_1(t) \cdot (\text{}^T\text{FR}_t-1)^2 + Q_2(t) \cdot (\text{}^T\text{FR}_t-1) + Q_3(t), \quad \text{in which} \end{aligned} \quad (25)$$

$$\begin{aligned}
B_1({}^TFR_{t-1}, t) &= [(1-\theta) + (\theta + Q_1(t+1)) \cdot (\sigma_1^2 + \mu_1(t)^2)] \cdot ({}^TFR_{t-1})^2 \\
B_2({}^TFR_{t-1}, t) &= -2 \cdot (\theta + Q_1(t+1)) \cdot (\sigma_1^2 + \mu_1(t)^2) \cdot ({}^TFR_{t-1})^2 - \\
&\quad [2(\theta + Q_1(t+1)) \cdot (\mu_1(t) \cdot \mu_2(t) + \sigma^2/\exp(\alpha + \beta + \eta)) + \mu_1(t) \cdot Q_2(t+1)] \cdot ({}^TFR_{t-1}) \\
B_3({}^TFR_{t-1}, t) &= (\theta + Q_1(t+1)) \cdot (\sigma_1^2 + \mu_1(t)^2) \cdot ({}^TFR_{t-1})^2 + \\
&\quad [2(\theta + Q_1(t+1)) \cdot (\mu_1(t) \cdot \mu_2(t) + \sigma^2/\exp(\alpha + \beta + \eta)) + \mu_1(t) \cdot Q_2(t+1)] \cdot ({}^TFR_{t-1}) + \\
&\quad (\theta + Q_1(t+1)) \cdot (\sigma_2^2 + \mu_2(t)^2) + \mu_2(t) \cdot Q_2(t) + Q_3(t+1)
\end{aligned}$$

Accordingly, the sequence $\{^*k_t(\cdot): t \in \Omega_1 - \{t^*-1\}\}$ is uniquely determined under the condition such that $B_1({}^TFR_{t-1}, t) > 0$ for all $t \in \Omega_1 - \{t^*-1\}$: that is, for each $t \in \Omega_1 - \{t+1\}$,

$$^*k_t({}^TFR_{t-1}) = \frac{M_1(t)}{M_3(t)} + \frac{M_2(t)}{M_3(t)} \times ({}^TFR_{t-1})^{-1}, \quad \text{in which} \quad (26)$$

$$\begin{aligned}
M_1(t) &= (\theta + Q_1(t+1)) \cdot (\sigma_1^2 + \mu_1(t)^2), \\
M_2(t) &= \mu_1(t) \cdot Q_2(t+1)/2 + (\theta + Q_1(t+1)) \cdot (\mu_1(t) \cdot \mu_2(t) + \sigma^2/\exp(\alpha + \beta + \eta)) \quad \text{and} \\
M_3(t) &= (1-\theta) + M_1(t).
\end{aligned}$$

In order to specify $^*k_t(\cdot)$, which depend only on the unknown $Q_1(t+1)$ and $Q_2(t+1)$ (not $Q_3(t+1)$), we need only to specify the following backward recursive equations, obtained by substituting $^*k_t({}^TFR_{t-1})$ into (25): for each $t \in \Omega_1 - \{t+1\}$,

$$\begin{aligned}
Q_1(t) &= \frac{(1-\theta) \cdot (\theta + Q_1(t+1)) \cdot (\sigma_1^2 + \mu_1(t)^2)}{(1-\theta) + (\theta + Q_1(t+1)) \cdot (\sigma_1^2 + \mu_1(t)^2)} \\
&\quad \text{with the boundary condition } Q_1(t^*-1) \text{ given in (24) and} \\
Q_2(t) &= \frac{(1-\theta) \cdot [\mu_1(t) \cdot Q_2(t+1) + 2(\theta + Q_1(t+1)) \cdot (\mu_1(t) \cdot \mu_2(t) + \sigma^2/e^{\alpha + \beta + \eta})]}{(1-\theta) + (\theta + Q_1(t+1)) \cdot (\sigma_1^2 + \mu_1(t)^2)} \\
&\quad \text{with the boundary condition } Q_2(t^*-1) \text{ given in (24).} \quad (27)
\end{aligned}$$

Then, each of these equations produces such a backward solution as $\{Q_1(t^*-1), Q_1(t^*-2), \dots, Q_1(1), Q_1(0)\}$ and $\{Q_2(t^*-1), Q_2(t^*-2), \dots, Q_2(1), Q_2(0)\}$, and hence the above uniqueness condition such that for all $t \in \Omega_2 - \{t^*-1\}$, $B_1({}^TFR_{t-1}, t) > 0$, is clearly satisfied because $Q_1(t^*-1) > 0$ and then $Q_1(t) > 0$.

Thus, we complete the mathematical induction argument and finally demonstrate that the optimal functional equation (25) has a solution of the quadratic form with $Q_1(t)$, $Q_2(t)$ and $Q_3(t)$ satisfying the recurrence relations (27). Further, $V({}^TFR_{t-1}, t)$ turns out to be quadratic, subject to ${}^TFR_{t-1} \neq 0$. As in the time-invariant part, there is no loss of generality in adopting a pragmatic rule that sets $^*k_t({}^TFR_{t-1}) = 0$ in the case of ${}^TFR_{t-1} = 0$, for ensuring the uniqueness of the optimal sequence $\{^*k_t({}^TFR_{t-1}); t \in \Omega_1\}$.

VI. Comments

Our heuristic optimisation procedure over an infinite horizon Ω , with respect to k_t is classified into two parts - designing optimally the optimal non-stationary control policy $^*k_t(\cdot)$ over Ω_1 and the optimal stationary control policy $^*k(\cdot)$ over Ω_2 : that is, for each Ω ,

$$^*k_t = ^*k_t({}^TFR_{t-1}) \cdot 1_{\Omega_1} + ^*k({}^TFR_{t-1}) \cdot 1_{\Omega_2}.$$

On the other hand, substituting $^*k_t(^TFR_{t-1})$ and $^*k(^TFR_{t-1})$, as defined in (23) and (26), and (19) respectively, into the threshold contribution ratio formula (13) yields the following optimal threshold contribution ratio formula: that is, for each $t \in \Omega$,

$$\begin{aligned} {}^T CR_t^* &= NR_t - [^*k_t(^TFR_{t-1}) \cdot 1_{\Omega_1} + ^*k(^TFR_{t-1}) \cdot 1_{\Omega_2}] \cdot [^TFR_t - 1] \\ (28) \quad &= NR_t - \left[\frac{M_1(t)}{M_3(t)} \cdot 1_{\Omega_1} + \frac{M_1}{M_3} \cdot 1_{\Omega_2} \right] \cdot [^TFR_t - 1] + \left[\frac{M_2(t)}{M_3(t)} \cdot 1_{\Omega_1} + \frac{M_2}{M_3} \cdot 1_{\Omega_2} \right] \\ &= NR_t^* - [\lambda_t \cdot 1_{\Omega_1} + \lambda \cdot 1_{\Omega_2}] \cdot [^TFR_t - 1], \end{aligned}$$

where,

$$NR_t^* = NR_t + \left[\frac{M_2(t)}{M_3(t)} \cdot 1_{\Omega_1} + \frac{M_2}{M_3} \cdot 1_{\Omega_2} \right], \quad \lambda_t = \frac{M_1(t)}{M_3(t)}, \quad \lambda = \frac{M_1}{M_3} \quad \text{and} \quad 0 < \lambda_t, \lambda < 1$$

Accordingly, the optimal threshold control response corresponding to ${}^T CR_t^*$ is recursively generated with time by the optimal threshold dynamic equation: for each $t \in \Omega$ and initially given ${}^TFR_{0-1} = {}^TFR_{0-1}^*$,

$$\begin{aligned} {}^TFR_{t+1}^* - 1 &= [n_1(t) \cdot \frac{M_3(t) - M_1(t)}{M_3(t)} \cdot 1_{\Omega_1} + s_1(t) \cdot \frac{M_3 - M_1}{M_3} \cdot 1_{\Omega_2}] \cdot [^TFR_t^* - 1] \\ &\quad + \left[n_1(t) \cdot \frac{M_2(t)}{M_3(t)} + n_2(t) \right] \cdot 1_{\Omega_1} + \left[s_1(t) \cdot \frac{M_2}{M_3} + s_2(t) \right] \cdot 1_{\Omega_2}, \end{aligned} \quad (29)$$

where $n_1(t)$ and $n_2(t)$ have been defined in (7) and $s_1(t)$ and $s_2(t)$ have been defined in (14).

Bowers et al. (1979) describe the spread method as normal cost plus amortisation over a moving term. In comparison, the proposed optimal control approach provides a similar type of formula, with the spread method as given in (28) but producing an optimal normal cost as well as optimal amortisation, which has been adjusted in anticipation of the future random rates of return. Thus, “ $\lambda_t \cdot 1_{\Omega_1} + \lambda \cdot 1_{\Omega_2}$ ” can be interpreted as a new amortisation parameter spreading forward the surpluses/deficits (i.e. ${}^TFR_t - 1$) in the light of optimal control theory, and also “ NR_t^* ” can be interpreted as a new normal cost ratio adjusting the classical actuarial normal cost ratio NR_t from the viewpoint of optimal control theory. In these respects, “ $\lambda_t \cdot 1_{\Omega_1} + \lambda \cdot 1_{\Omega_2}$ and NR_t^* ” each could be called a threshold optimal controller.

Comparing the spread funding formula (6) based on the classical, static actuarial practice with the above funding formula (28) derived from optimal control theory, we note that the newly defined spread parameter $\lambda_t \cdot 1_{\Omega_1} + \lambda \cdot 1_{\Omega_2}$ plays a similar role to k_t on the grounds that $0 < k_t$, $\lambda_t \cdot 1_{\Omega_1} + \lambda \cdot 1_{\Omega_2} < 1$ for each $t \in \Omega$ and that it spreads the recognised investment gains/losses over subsequent

periods, whereas the term $\frac{M_2(t)}{M_3(t)} \cdot 1_{\Omega_1} + \frac{M_2}{M_3} \cdot 1_{\Omega_2}$ is additionally required as an adjustment to NR_t ,

for the realisation of our funding objective (i.e. the on-going balanced stability in contribution ratios and funding ratios against their own targets).

VII. Numerical Illustrations and Conclusions

Here, we illustrate numerically the control effects with respect to the future movement of investment returns: in particular, in terms of contribution risk and solvency risk and their trade-offs.

1. Assumptions

- Actuarial Assumptions

: $\exp(\alpha) = 1.02$, $\exp(\beta) = 1.02$ and $\exp(\eta) = 1.06$;

- Projection Assumptions

: initial funding ratio $FR_0 = 80\%$ and 120%

: infinite control horizon: $t \in \Omega$;

: value of $\theta = 10\%$, 30% , 50% , 70% and 90% ;

: i_{t+1} is governed by equation (4) where $i_e = 0.06$, $\phi = 8$, $\varpi = 24\pi$ and $\sigma = 30\% \cdot i_e$; and

: threshold time value: $t^* = 59$ (i.e. $\Omega_1 = \{t: 0, 1, 2, \dots, 59\}$ and $\Omega_2 = \{t: 60, 61, 62, \dots\}$).

2. Concluding Remarks

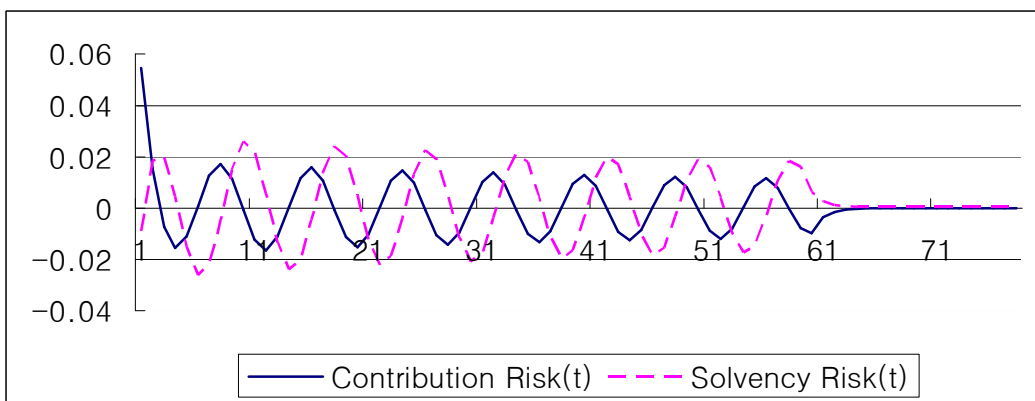
The following table shows the threshold spreading values (i.e. $\lambda_t \cdot 1_{\Omega_1} + \lambda \cdot 1_{\Omega_2}$) and the threshold adjustments of NR_t (i.e. $NR_t^* - NR_t = \frac{M_2(t)}{M_3(t)} \cdot 1_{\Omega_1} + \frac{M_2}{M_3} \cdot 1_{\Omega_2}$), respectively. The figures of Table 1 lead to the conclusion that, in order to minimise (contribution and solvency) risk, it is necessary to control optimally both the procedure of spreading surplus/deficit and the procedure of changing the current actuarial assumptions. In the light of generally accepted actuarial funding practice, the period of amortisation would be suggested to be somewhat shorter or equal to the expected average future-working lifetime of the current employees. In this respect, Table 1 clearly shows that placing more weight on contribution stability rather than on benefit security could be thought to be more practically admissible. Of course, it is to be determined whether the scheme actuary should allow faster or slower recognition of actuarial surpluses/deficits (i.e. $\theta \rightarrow 100\%$ or 0%) by balancing the conflicting interests between the scheme's sponsor and members.

<Table 1> Values of Threshold Optimal Controllers

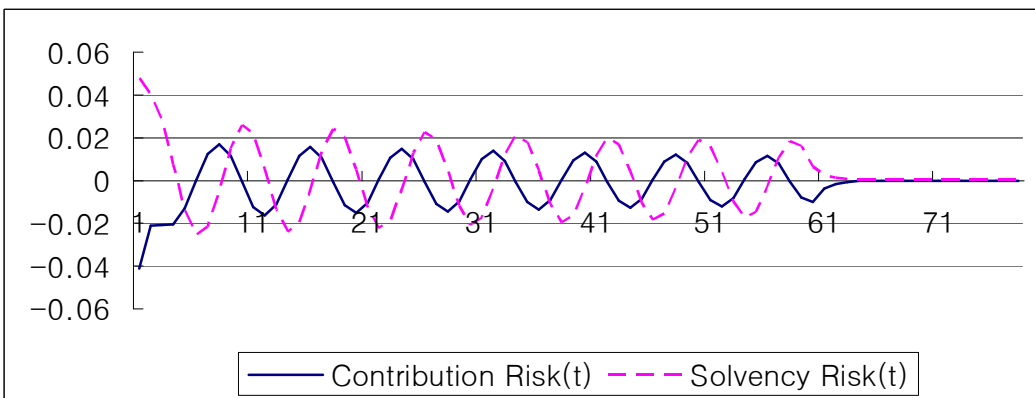
θ	$\lambda_t \cdot 1_{\Omega_1}$			$\lambda \cdot 1_{\Omega_2}$	$(NR_t^* - NR_t) \cdot 1_{\Omega_1}$			$(NR_t^* - NR_t) \cdot 1_{\Omega_2}$
	Range [Min, Max]	MEAN	STD.		Range [Min, Max]	MEAN	STD.	
0.1	[0.29178, 0.30507]	0.29865	0.00408	0.30068	[-0.00458, 0.00547]	0.00057	0.00295	0.000295
0.3	[0.48056, 0.49461]	0.48776	0.00423	0.48850	[-0.00720, 0.00810]	0.00062	0.00461	0.000288
0.5	[0.62226, 0.63458]	0.62856	0.00361	0.62878	[-0.00848, 0.00935]	0.00063	0.00554	0.000286
0.7	[0.75824, 0.76729]	0.76287	0.00263	0.76286	[-0.00996, 0.01080]	0.00064	0.00618	0.000284
0.9	[0.90949, 0.91339]	0.91149	0.00113	0.91144	[-0.01101, 0.01180]	0.00064	0.00665	0.000283

<Figures 1 & 2> given below show the resulting values (i.e. $E[{}^T\text{CR}_t^* - \text{NR}_t | {}^T\text{FR}_{t-1}]$ and $E[{}^T\text{FR}_{t+1}^* - 1 | {}^T\text{FR}_t^* - 1]$), illustrating the (control action and control response) volatilities against their own targets: In order to emphasize this respect, $E[{}^T\text{CR}_t^* - \text{NR}_t | {}^T\text{FR}_{t-1}]$ and $E[{}^T\text{FR}_{t+1}^* - 1 | {}^T\text{FR}_t^* - 1]$ are denoted as (control) contribution risk(t) and (control) solvency risk(t), respectively. These figures clearly show that there is a time-delayed trade-off between contribution risk(t) and solvency risk(t) under the non-stationary circumstances and also that the trade-off decreases as we move from the non-stationary to the stationary case. And also, as illustrated in <Figure 1 & 2>, the initial conditions have more effect on the phase of the cycles than on the frequency or the amplitude.

<Figure 1> In a case of Under-funding $\text{FR}_0 = 80\%$ & $\theta = 50\%$



<Figure 2> In a case of Over-funding $\text{FR}_0 = 120\%$ & $\theta = 50\%$



Consequently, we believe that the classical spread formula (6) is not adequate for achieving the purpose of pension funding principally because of the variability inherent in the rates of investment return. So, we would like to suggest that the actuarial profession explores methodologies (such as those introduced in this paper) for reflecting in advance the estimates of the future demographic and economic experience in the pension funding mechanism, rather than depend on fixed valuation assumptions and realised accounting results on the ground that the pension funding management has become an increasing concern for governments in developed as well as developing countries. In

this aspect, we hope that this paper can provide a turning point of re-diagnosing the currently popular funding management.

References

- Bertsekas, D.P., *Dynamic Programming and Stochastic Control*, New York: Academic Press, 1976.
- Bowers, B.N., Hickman, J.C., and Nesbitt, C.J., "The Dynamics of Pension Funding: Contribution Theory," *Transactions of the Society of Actuaries* 31, 1979, pp.93~122.
- Dufresne, D., "Moments of Pension Contributions and Fund Levels when Rates of Return are Random," *Journal of the Institute of Actuaries*, 115, 1988, pp.535~544.
- Haberman, S. and Sung, J.H., "Dynamic Approaches to Pension Funding," *Insurance: Mathematics and Economics*, 15, 1994, pp.151~162.
- _____, S. and Sung, J.H., "Dynamic Programming Approach to Pension Funding: The Case of Incomplete State Information," *ASTIN Bulletin*, Vol. 32, No. 1, 2002, pp.129~142.
- _____, S. and Sung, J.H., "Optimal Pension Funding Dynamics over Infinite Control Horizon When Stochastic Rates of Return are Stationary," *Insurance: Mathematics and Economics*, 2004, to appear.
- Loades, D.H., "Instability in Pension Funding," *Transactions of 24th International Congress of Actuaries*, 2, 1992, pp.137~154.
- McGillem, C.D. and Cooper, G.R., *Continuous and Discrete Signal and System Analysis*, 3rd Edition, Philadelphia: Sanders College Publishing, 1991.
- Owadally, M.I. and Haberman, S., "Pension Fund Dynamics and Gains/Losses due to Random Rates of Investment Return," *North American Actuarial Journal*, 3 (3), 1999, pp.105~117.
- Sung, J.H., "Optimal Control Approach to Spread Pension Funding When Rates of Return are Non-Stationary," *Korean Insurance Journal*, Vol. 66, 2003, pp.29~54.
- Tong, H. And Lim, K.S., "Threshold Autoregression, Limit Cycles and Cyclical Data," *Journal of Royal Statistical Society, Series B42*, 1980, pp.245~292.
- Whittle, P., *Optimization over Time: Dynamic Programming and Stochastic Control. Vol. I*, Chichester: John Wiley & Sons, 1982.

<Abstract>

This study follows up the earlier works of Haberman & Sung (1994, 2002, 2004) who have adopted a dynamic approach to pension funding in order to control and harmonize simultaneously the contribution risk and the solvency risk, based on a linear stochastic dynamic system with a quadratic optimisation criterion (LQP problem). In contrast to these earlier works, we here consider funding plans for defined benefit mature pension schemes where the spread method is used to eliminate the solvency surpluses and deficiencies evaluated in relation to their own expecting funding targets. Moreover, we consider the infinite-time, non-stationary LQP optimisation control problem due to trending non-stationary rates of return, which is a theoretical extension of Sung (2003) dealing with deterministic approach. We note that it is insoluble but propose a heuristic optimisation procedure for solving this problem and then illustrate with a specific numerical projections of contribution and solvency risks.

Key words: pension funding, infinite horizon LQP problem, spread method, dynamic programming, non-stationary stochastic process, solvency risk, contribution risk, heuristic projections.

K C I