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Optimization of Dynamic Guaranteed Minimum Return, Investment And Reinsurance Strategy By Balancing the Risks And Benefit of Both Insurers And Consumers*

보험회사와 소비자의 이익 및 리스크를 고려한 동적 최저보증 수익, 투자전략, 재보험전략의 최적화에 관한 연구

Hong Mao** · James M. Carson*** · Krzystof M. Ostaszewski****

In this article, we model and discuss determination of optimal minimum guaranteed rate of return, as well as optimal investment and reinsurance strategies of universal life insurance by minimizing both the investment risk and the risk of obtaining guaranteed return, with the constraint of surplus larger than a prescribed constant. We also discuss the application of dynamic programming in finding dynamic solutions to these optimization problems. We analyse the affect of the change of the risk-free interest rate, the age of insured, the cost of reinsurance, and mortality on optimal solutions. Our results indicate that changes in the insured age, in the risk-free interest rate (when risk-free interest rate takes high value), and of mortality will not materially affect the optimal value of minimum guaranteed return rate, investment and reinsurance strategies except for the situation when mortality decreases. However, changing these parameters will affect the sum of the volatilities of investment and minimum guaranteed return rate and the surplus of the insurer. The results also indicate that the optimal and sub-optimal minimum guarantee return rates are very low when risk-free interest rate is very low ($r_f = 0.1\%$).

Key words: Life insurance, Universal life, Minimum interest rate guarantee, Optimal risk management
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** Shanghai Second Polytechnic University(maohong@sspu.edu.cn), The first author
*** Daniel P. Amos Distinguished Professor of Insurance, Terry College of Business, University of Georgia(jcarson@uga.edu)
**** Department of Mathematics, Illinois State University(krzysio@ilstu.edu), Corresponding author

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I. Introduction

Universal life insurance is a very popular life insurance product in the U.S and other countries as a whole because of its flexibility and providing the policyholder with minimum guarantee return so as to decrease the risk taken by policyholders. Currently, some insurance companies in China have begun to sell this product. However, because interest rates in developed economies and also in developing economies are so low, life insurance companies are concerned about the cost of those guarantees.

An important thing is that the minimum guarantee implicitly represents short positions in financial derivatives together with other elements of optimality such as bonus distribution schemes and surrender possibilities and as such are liabilities which constitute a potential hazard to company solvency(Mahayni and Schlogl, 2003). How to determine the optimal guarantee return rate? If it is too high, it will increase the insolvency risk of insurers, but if it is too low, it will reduce the policyholder's benefits so as to decrease the motive for consumer to buy life insurance policies and further damage the company solvency. The valuation of life products with minimum guaranteed return rate has been discussed in a lot of literature. Persson and Aase (1997), Grosen(1997), Briys and Varenne(1997) have proposed the valuation of life products with the minimum guaranteed return. Bacinello and Ortu(1993), Nielsen and Sandmann(1995), Boyley and Hardy(1997), Grosen and Joergensen(1997) and Bacinello(2001) studies unit-linked contracts with minimum return guarantee. Moeller (2001) discussed risk-minimizing hedging strategies for a general unit-linked life insurance contract driven by a Markov jump process and a claim process from non life insurance where the claim size distribution is affected by a traded price index. Milevsky and Salisbury(2005) discussed the financial valuation of guaranteed minimum withdraw benefit for variable annuity. However, as Mahayni and Schlogl(2003) pointed out:“ the existing literature is mainly concerned with the correct valuation of insurance policies, i.e. the pricing of the option component by standard Black/Scholes−type dynamic arbitrage arguments”. Risk management of insurance products must consider and balance benefits and risks of both consumers’ and insurers’. In our article, minimum guaranteed return rate is seen as a kind of price of insurance policies. Minimum guaranteed return rate g is determined based on the objective to minimize the sum of volatilities of investment and minimum guaranteed return rate, at the same time, satisfy a certain constraint, that is, the surplus rate of insurance companies is equal to or larger than a constant. The determination of the premium is still based on the equivalence principle, that is, the net premium is equal to the total claim payment due to the death events. Therefore, for universal life insurance, it is necessary to determine two different prices, one is premium not including the factor of investment and another is minimum guaranteed return rate which is necessary to consider the hedging factor of investment risk. Since it is general practice in actuarial science on determining the life insurance premium without considering investment factor, here we will not discuss it. We mainly focus on discussing how to determine optimal minimum guaranteed rate. We also consider the hedge of the investment return to the underwriting risk and to the payment of minimum guaranteed return and assume that return rates of risk assets invested are Vasicek(1977) stochastic processes. We transfer the Vasicek models into Gaussian stochastic process so that we can easily formulation the equation of calculating the volatility of the minimum guarantee return rate.

In this article, we apply similar method to that of dynamic mean−variance(M−V) model[1]. But the difference between our approach and those in existing literature is

1) Since Markowitz(1952) proposed mean−variance(M−V) portfolio selection, there is a lot of literature to study and extend the Markowitz’s M−V model (1) to formulate dynamic M−V models; (2) to combine stochastic optimal control theory to derive the expression ofn the efficient strategy and efficient frontier in closed forms; (3) to extend the dynamic M−V model to cases with a variety of more realistic conditions; and to adopt the dynamic M−V model to study the ALM problem(Yao, Lai and Li, 2013).

that we use minimizing the sum of volatilities of investment return rate and minimum guaranteed return rate instead of minimizing the volatility of the terminal surplus as an objective function.

The model proposed in this article mainly has two advantages. The first one is that the determination of minimum guaranteed return rate g considers both the insurer's and policyholder's risks and benefits. Therefore, it can decrease insolvency risk and at the same time raise the motive for consumer to buy this universal life insurance because of minimum claim risk and insolvency risk of insurance companies. The second advantage is that the optimal solutions can be dynamic with time. Therefore, it is more truly reflecting the real situation of insurance companies.

The remaining part of this article is organized as follows. In the next section, several assumptions are discussed, a general valuation model is proposed, and determination of optimum g is discussed. In the third section of this article an example is given to illustrate its application and numerical analysis is carried out. Last section gives our conclusions.

1. Valuation model of universal life insurance with level premium paid at the beginning of each year

1.1. Assumptions

Assume that the life product is a universal life insurance product and the term of each policy is $T_{0.}$ Each life has the same death distribution and the death events are independent of each other. The death probability of the insured aged x who is alive at $x+t-1$ but dead at $x+t$ is expressed as q_{x+t}.

Assume that the rate of return of investment portfolio can be expressed as the following stochastic differential equation(because the return rate of investment may have negative values, we use Vasicek model):

$$dr = a(b_r - r)dt + \sigma_r dz_r \qquad (1)$$

where dz_r is a standard Wiener process, σ_r is the standard deviation of return rate of investment portfolio, b_r is the equilibrium return rate of investment portfolio of long term, $b_r - r$ is the gap between its current rate of return and its long−run equilibrium level and a is a parameter measuring the speed at which the gap diminish. Assuming that the diversified portfolio consists of one risk−free asset and n types of risky investments, the fraction invested in i−th risky investment is α_i, $i = 1,2,\ldots,n$, the fraction invested in risk−free asset is α_{n+1} and $\sum_{i=1}^{n+1}\alpha_i = 1$, then the return on risky asset i follows equation.

$$dr_i = a_i(b_i - r_i)dt + \sigma_i dz_i \qquad (2)$$

and a portfolio return is $r = \sum_{i=1}^{n}\alpha_i r_i$. The differential of the portfolio return is

$$\sum_{i=1}^{n}\alpha_i dr_i = \sum_{i=1}^{n}\alpha_i\left(a_i(b_i - r_i)dt + \sigma_i dz_i\right) = \sum_{i=1}^{n}\alpha_i a_i(b_i - r_i)dt + \sum_{i=1}^{n}\alpha_i\sigma_i dz_i \qquad (3)$$

If $a_i = a$, then

$$b_r = \sum_{i=1}^{n}\alpha_i b_i \qquad (4)$$

If the correlation between z_i and z_j is ρ_{ij}, then the variance of the portfolio return is

$$V_t\left(\sum_{i=1}^{n}\alpha_i\sigma_i dz_i\right) = \left(\sum_{i=1}^{n}\alpha_i^2\sigma_i^2 + \sum_{i=1}^{n}\sum_{\substack{j=1 \\ i\neq j}}^{n}\alpha_i\alpha_j\sigma_i\sigma_j\rho_{ij}\right)dt$$

and the standard deviation is

$$\sigma_r = \sqrt{\sum_{k=1}^{n}\alpha_k^2\sigma_k^2 + \sum_{i=1}^{n}\sum_{\substack{j=1 \\ i\neq j}}^{n}\alpha_i\alpha_j\rho_{ij}\sigma_i\sigma_j} \tag{5}$$

Based on Momon(2004), we know that the expected return rate of risky investment portfolio is

$$E(r) = \mu_r(t) = E\sum_{i=1}^{n}\alpha_i r_i = \sum_{i=1}^{n}\alpha_i\mu_i(t) = \sum_{i=1}^{n}\alpha_i e^{-a_i t}\left(r_i(0) + b_i\left(e^{a_i t} - 1\right)\right), \tag{6}$$

where $\mu_i(t)$ is the expected return rate of i−th risky asset at time t in real world measure and $r_i(0)$ is the return rate of i−th risky investment at time $t=0$. And we define

$$\sigma_r^2(t) = Var\left(r\right) = Var\left(\sum_{i=1}^{n}\alpha_i r_i\right) = \sum_{k=1}^{n}\alpha_i^2\sigma_i^2(t) + \sum_{i=1}^{n}\sum_{\substack{j=1 \\ i\neq j}}^{n}\alpha_i\alpha_j\sigma_{ij}(t), \tag{7}$$

where $\sigma_i^2(t)$ is the volatility of the return rate of i−th risky asset in real world measure, $\sigma_i^2(t) = \sigma_i^2\left(\frac{1-e^{-2a_i t}}{2a_i}\right)$ and $\sigma_{ij}(t)$ is covariance between i−th risky asset and j−th asset in real world measure,

$$\sigma_{ij}(t) = e^{A_{it}+\frac{1}{2}B_{it}+A_{jt}+\frac{1}{2}B_{jt}+Y_t} - e^{A_{it}+\frac{1}{2}B_{it}+A_{jt}+\frac{1}{2}B_{jt}} \tag{8}$$

where

$$A_{it} = \frac{r_{i0} - b_i}{a_i}(1 - e^{-a_i t}) - b_i t \qquad B_{it} = \frac{\sigma_i^2}{2a_i^3}(2a_i t - 3 + 4e^{-a_i t} - e^{-2a_i t})$$

$$A_{jt} = \frac{r_{j0} - b_j}{a_j}(1 - e^{-a_j t}) - b_j t \qquad B_{jt} = \frac{\sigma_j^2}{2a_j^3}(2a_j t - 3 + 4e^{-a_j t} - e^{-2a_j t})$$

and

$$Y_t = \frac{1}{a_i + a_j}\left(\begin{array}{l} \frac{\sigma_i \sigma_j}{a_i + a_j}\left(\frac{1}{a_i^2}(a_i t + e^{-a_i t} - 1) + \frac{1}{a_j^2}(a_j t + e^{-a_j t} - 1) \right) \\ -\frac{\sigma_i \sigma_j}{a_i a_j}(e^{-a_i t} - 1)(e^{-a_j t} - 1) \end{array} \right).$$

For the proof of equation (8), please see Mao et al.(2012). Since $r_i(t) \sim N(\mu_i(t), \sigma_i^2(t))$, $r_i(t)$ also satisfies the stochastic differential equation: $dr_i(t) = \mu_i(t)dt + \sigma_i(t)dz_i$ and the return rate of investment portfolio $r(t)$ satisfies

$$dr(t) = \mu_r'(t)dt + \sigma_r(t)dz_r \tag{9}$$

where $\mu_r'(t) = \mu_r(t) + \alpha_{n+1} r_f$.

Assume that mortality is independent of the investment return, and follows the Gompertz–Makeham distribution(Milevsky, 2006), with the instantaneous force of mortality(IFM) given as:

$$\lambda(x) = \delta + \frac{1}{b}e^{(x-m)/b},\ t \geq 0 \tag{10}$$

where m is the modal value of future lifetime and b is the dispersion coefficient. According to the equation (10), the instantaneous force of mortality is a constant δ plus a time-dependent exponential curve. The constant δ aims to capture the component of the death rate that is attributable to accidents, while the exponentially increasing portion reflects nature death. The conditional probability of survival under this Gompertz−Makeham IFM curve is equal to

$$
\begin{aligned}
{}_t p_x &= \exp\left\{-\int_x^{x+t}\left(\delta+\frac{1}{b}e^{(s-m)/b}\right)ds\right\}= \\
&= \exp\left\{-\delta t+b(\lambda(x)-\delta)(1-e^{t/b})\right\}
\end{aligned}
\tag{11}
$$

Assuming the surplus of the insurer at time t, $X(t)$, satisfies the following stochastic differential equation:

$$
\begin{aligned}
dX(t) &= \left(q_{x+t}\operatorname{Re}(1-\eta)+X(t)\mu_r{}'(t)-\operatorname{Re}q_{x+t}\max(e^{gt},e^{\mu_r{}'t}\right)dt+ \\
&+X(t)\sigma_r(t)dz-\operatorname{Re}q_{x+t}\sigma_g(t)dz_1,
\end{aligned}
\tag{12}
$$

with boundary condition $X(0)=1$, where Re is the proportion of retention of reinsurance and η is the rate of reinsurance cost.

1.2. Valuation Models for Static Solutions

The stochastic differential equation (12) has the unique solution based on Ito's Lemma:

$$
\begin{aligned}
X(t) &= X_0+\int_0^t X(s)\mu_r{}'(s)\,ds+\int_0^t X(s)\sigma_r(s)\,dz- \\
&-\int_0^t \operatorname{Re}(q_{x+s}\max(e^{gs},e^{\mu_r{}'s})\,ds+q_{x+s}\sigma_g(s)dz_1)
\end{aligned}
\tag{13}
$$

$$
\begin{aligned}
E^P\left(X(t)\right) &= E^P\left(X_0+\int_0^t X(s)\mu_r{}'(s)\,ds+\int_0^t X(s)\sigma_r(s)\,dz\right)+ \\
&+E^P\left(\int_0^t \operatorname{Re}(q_{x+s}\max(e^{gs},e^{\mu_r{}'s})\,ds+q_{x+s}\sigma_g(s)dz_1)\right)
\end{aligned}
\tag{14}
$$

where $E^P(\square)$ is the expectation operator under the real world measure and g is minimum guarantee return rate. By the constraint condition that $E^P(X(T_0)-X(0)) \geq C$, we have

$$E^P\left(X(T_0)-X(0)\right)=E^P\left(X_0+\int_0^{T_0}X(s)\mu_r{}'(s)\,ds+\int_0^{T_0}X(s)\sigma_r(s)\,dz\right)-$$
$$-E^P\left(\int_0^{T_0}\mathrm{Re}(q_{x+s}\max(e^{gs},e^{\mu_r{}'s})\,ds+q_{x+s}\sigma_g(s)dz_1)\right)-X(0)\geq C \tag{15}$$

where $\mu_r{}'(t)=\mu_r(t)+\alpha_{n+1}r_f$, $\sigma_r(t)=\sqrt{\sum_{k=1}^{n}\alpha_k^2\sigma_k^2(t)+\sum_{i=1}^{n}\sum_{\substack{j=1\\i\neq j}}^{n}\alpha_i\alpha_j\sigma_{ij}(t)}$, and $q_{x+t}={}_{t-1}p_x-{}_tp_x$ is the mortality rate at time t, x is the age the insurance policy is issued, T_0 is the maturity of insurance contracts and

$$\sigma_g^2(t)=\int_{-\infty}^{g}(g(t)-E(x(t)))^2f(x(t))dx+\int_{g}^{+\infty}(x(t)-E(x(t)))^2f(x(t))dx$$
$$=\int_{-\infty}^{g}(g(t)-\mu_r{}'(t))^2f(x(t))dx+\int_{g}^{+\infty}(x(t)-\mu_r{}'(t))^2f(x(t))dx \tag{16}$$
$$=(g(t)-\mu_r{}'(t))^2\Phi\left(\frac{g(t)-\mu_r{}'(t)}{\sigma_r(t)}\right)+\sigma_r^2(t)$$

Letting the objective function be

$$\min\left(\sum_{t=1}^{T_0}\sigma^2(t)\right)^{\frac{1}{2}}=$$
$$=\left(\sum_{t=1}^{T_0}(q_{x+t})^2(\sigma_r^2+\sigma_g^2)=\sum_{t=1}^{T_0}(q_{x+t})^2\left((g(t)-\mu_r{}'(t))^2\Phi\left(\frac{g(t)-\mu_r{}'(t)}{\sigma_r(t)}\right)+2\sigma_r^2\right)\right)^{\frac{1}{2}} \tag{17}$$
$$\mathrm{st}:E^P(X(T_0)-X(0))\geq C$$

Write the Lagrange equation of objective function (17) as:

$$L(g)=\sum_{t=1}^{T_0}(q_{x+t})^2\left((g(t)-\mu_r(t))^2\Phi\left(\frac{g(t)-\mu_r(t)}{\sigma_r(t)}\right)+2\sigma_r^2(t)\right)+\lambda(E(X(T)) \tag{18}$$

The one order conditions of Lagrange equation with respect to $g,\alpha_i,i=1,2,...n$, Re can be written as:

$$\begin{aligned}&\frac{\partial L}{\partial\alpha_i}=\sum_{t=1}^{T_0}q_{x+t}^2\left(g(t)-\mu_r{}'(t)\right)\left(2\Phi\left(\frac{g(t)-\mu_r{}'(t)}{\sigma_r(t)}\right)\frac{\partial\mu_r{}'}{\partial\alpha_i}+\left(g(t)-\mu_r{}'(t)\right)\frac{\partial\Phi}{\partial\mu_r{}'}\frac{\partial\mu_r}{\partial\alpha_i}\right)\\&+\lambda\left(\begin{aligned}&\frac{\partial}{\partial\alpha_i}E^P\left(X_0+\int_0^T X(s)\mu_r{}'(s)ds+\int_0^T X(s)\sigma_r(s)dz\right)-\int_0^T \mathrm{Re}\,q_{x+s}\frac{\partial}{\partial\mu_r{}'}\max\left(e^{gs},e^{\mu_r{}'s}\right)\frac{\partial\mu_r}{\partial\alpha_i}ds\\&-\frac{\partial}{\partial\mu_r{}'}\int_0^T \mathrm{Re}\,q_{x+s}\left(g(t)-\mu_r{}'(t)\right)\left(2\Phi\left(\frac{g(t)-\mu_r(t)}{\sigma_r(t)}\right)\frac{\partial\mu_r{}'}{\partial\alpha_i}+\left(g(t)-\mu_r{}'(t)\right)\frac{\partial\Phi}{\partial\mu_r{}'}\frac{\partial\mu_r}{\partial\alpha_i}\right)dz\end{aligned}\right)\\&=0\end{aligned} \tag{19}$$

$$\begin{aligned}&\frac{\partial L}{\partial\lambda}=E^P\left(X_0+\int_0^{T_0}X(s)\mu_r{}'(s)ds+\int_0^{T_0}X(s)\sigma_r(s)dz\right)-\\&-E^P\left(\int_0^{T_0}\mathrm{Re}(q_{x+s}\max(e^{gs},e^{\mu_r{}'s})ds+{}_sq_x\sigma_g(s)dz_1)\right)-C=0\end{aligned} \tag{20}$$

$$\begin{aligned}&\frac{\partial L}{\partial g}=\sum_{t=1}^{T_0}q_{x+t}^2\left(g(t)-\mu_r(t)\right)\left(2\Phi\left(\frac{g(t)-\mu_r(t)}{\sigma_r(t)}\right)+\left(g(t)-\mu_r(t)\right)\frac{\partial\Phi}{\partial g}\right)\\&+\lambda\left(\begin{aligned}&\frac{\partial}{\partial g}E^P\left(X_0+\int_0^T X(s)\mu_r{}'(s)ds+\int_0^T X(s)\sigma_r(s)dz\right)-\int_0^T \mathrm{Re}\,q_{x+s}\frac{\partial}{\partial g}\max\left(e^{gs},e^{\mu_r s}\right)ds\\&-\frac{\partial}{\partial g}\int_0^T \mathrm{Re}\,q_{x+t}\left(g(t)-\mu_r(t)\right)\left(2\Phi\left(\frac{g(t)-\mu_r(t)}{\sigma_r(t)}\right)+\left(g(t)-\mu_r(t)\right)\frac{\partial\Phi}{\partial g}\right)dz_1\end{aligned}\right)\\&=0\end{aligned} \tag{21}$$

$$\begin{aligned}&\frac{\partial L}{\partial\mathrm{Re}}=2\sum_{t=1}^{T_0}\mathrm{Re}\,q_{x+t}{}^2\left((g(t)-\mu_r(t))^2\Phi\left(\frac{g(t)-\mu_r(t)}{\sigma_r(t)}\right)+2\sigma_r^2\right)\\&+\lambda\left(\begin{aligned}&\frac{\partial}{\partial\mathrm{Re}}E^P\left(X_0+\int_0^T X(s)\mu_r{}'(s)ds+\int_0^T X(s)\sigma_r(s)dz\right)\\&-E^P\left(\int_0^T(q_{x+s}\max(e^{gs},e^{\mu_r{}'s})ds+{}_sq_x\sigma_g(s)dz_1)\right)\end{aligned}\right)=0\end{aligned} \tag{22}$$

By solving the system of equations (19), (20), (21) and (22), we can get the optimal solution of g^* and optimal investment allocation strategy $(g, \alpha_1^*, \alpha_2^*, ..., \alpha_n^*, 1-\alpha_1^*-\alpha_2^*-...\alpha_n^*)$. However, it is impossible to get the explicit solutions of these four system equations by analysis methods. We use numerical method and optimization technique to get the approximated solutions.

1.3. Proper order dynamic programming models

In above section, we discuss the problem of finding the static solutions of optimal minimum guaranteed return rate, the proportions of risky assets invested and the proportion of reinsurance. In this section, we will discuss how to find the dynamic optimal solutions of these parameters by dynamic programming. Usually, the inverse order method is used to solve the problem of dynamic programming with the initial values of parameters given. However, since the boundary condition at last stage is given in our case, it is necessary for us to use proper order dynamic programming by dividing the total maturity time into several stage, and the duration for each stage is just one year, then, the process to find optimal solution is from first stage to the final stage. The objective function of the k−th stage with the constraint of the average surplus of each year being larger or equal to c can be written as:

$$\begin{aligned} &\min \sigma_k = \left(\sum_{i=1}^{k} \sigma^2(i) \right)^{\frac{1}{2}}, k = 1,2,...T_0 \\ &st: \ \frac{E(X(k) - X(0))}{k} \geq c \end{aligned} \tag{23}$$

where

$$\sigma^2(i) = (\mathrm{Re}\, q_{x+i})^2 \left((g(i) - \mu_r'(i))^2 \Phi\left(\frac{g(i) - \mu_r'(i)}{\sigma_r(i)} \right) + 2\sigma_r^2(i) \right)$$

Solving the objective functions of each stage, firstly, get the optimal solutions of the first stage, $g_1, \alpha_1^1, \alpha_2^1, \alpha_3^1$ and Re_1, secondly, put these optimal solutions into the objective function and solving it to get the optimal solutions in the second stage until the last stage where the optimal solutions $g_{T_0}, \alpha_1^{T_0}, \alpha_2^{T_0}, \alpha_3^{T_0}$ and Re_{T_o} are obtained.

2. Numerical Examples and Discussion

2.1. Without consideration of reinsurance

It is assumed that there are three kinds of investments: stocks, Treasury Bond, and Treasury Bill. The allocations of them in the investment portfolio are α_1, α_2 and α_3, respectively. We use Vasicek model to simulate the return rates of stocks and bonds. We use the data of S&P 500 index, the bonds[2]) from 1976 to 2009 to estimate the parameters of a_i, b_i and $\sigma_i, i=1,2$. We use the maximum likelihood estimation to find these parameters based on the formulas discussed in the book written by Gourieroux and Jasiak(2001) (Section 12.1.2). The estimated values of parameters are listed in Table 1. We estimate the risk-free return rate using the return rate of 3 months Treasure Bill from 1976 to 2009 and get $r_f=0.054$. $r_1(0)=0.2645$, $r_2(0)=-0.149$. We assume that the initial value of wealth $X_0=1$.

<Table 1> The values of parameters of S&P 500 stock market index and Government Bond estimated by maximum likelihood

MLE	S&P	Government Bond
estimation(a_i)	2.2183	0.7931
estimation(σ_i^2)	0.1318	0.0457
estimation (b_i)	0.1251	0.0959

We also assume $\lambda=0, m=86.34, b=9.5$ according to GoMa law[3]), the insured age at the time when the insurance policy issued is $x=40$, the insurance term is whole life, then $T_0=10$, the longest life span is 110(under

2) Sources: Stocks, Bonds, Bills, and Inflation Yearbook. The data is annually reported.
3) These were the best-fitting parameters to the unisex RP2000 mortality table, see Milevsky(2006).

the Gompertz−Makeham IFM curve, T ∞ is , but we assume that ${}_tP_x$ is zero when $t \geq 110$).

By considering the stochastic objective function with the constraint (23), and with the help of Monte Carlo simulation and optimization techniques, we get the optimized investment portfolio proportion, sub−optimal minimum guarantee rate and optimal reinsurance proportion when C takes value of 0,0.10 and 0.20 (please see Table 2, the solutions in lines 2, 4 and 6 are optimal solutions, while those in lines 3, 5 and 7 are sub−optimal solutions.).

<Table 2> The optimal and sub−optimal solutions of minimum guaranteed return rate and the proportions of investments without consideration of reinsurance

C	α_1*	α_2*	α_3*	g*	$\min\sqrt{\sum_{t=1}^{10}\sigma^2(t)}$	$E^*(X(10)-X(0))$
0	0.0262	0.0063	0.9674	0.0699	0.0011	0.0671
0	0.0182	0.0269	0.9546	0.0093	0.0009	0.0846
≥0.10	0.0262	0.0063	0.9674	0.0699	0.0014	0.1758
≥0.10	0.0182	0.0269	0.9546	0.0093	0.0014	0.2042
≥0.20	0.0256	0.0495	0.9249	0.0613	0.0025	0.2167
≥0.20	0.0294	0.0199	0.9507	0.0048	0.0017	0.3028

Our results (Table 2) show that the optimal minimum guaranteed return rate is very small when the total volatility reaches smallest, but the expected optimal surplus of the insurer is larger. It is not beneficial to the consumer. But it is conformed to the current situation, that is, very low interest rate due to the financial crisis occurred in 2008. We also get the sub−optimal minimum guaranteed return rate and corresponding investment strategy. Table 2 also shows that the bigger the value upper bound of constraint, the larger the optimal total volatility is.

2.2. With consideration of reinsurance

Reinsurance is an important tool to hedge the underwriting risk and investment risk and it can also be used as an important instrument of capital management when the reinsurance cost is not high. Table 3 lists the results of optimal solutions under the constraint value being zero.

<Table 3> The optimal and sub-optimal solutions of minimum guaranteed return rate and the allocations of investments under consideration of reinsurance

C	α_1*	α_2*	α_3*	g*	Re*	$\min\sqrt{\sum_{t=1}^{10}\sigma^2(t)}$	$E^*(X(10)-X(0))$
≥0	0.0966	0.0153	0.8881	0.0901	0.2637	0.000009	0.4075
≥0	0.1138	0.0019	0.8843	0.0169	0.2525	0.00006	0.6017

From Table 3, we find that the optimal and sub optimal surplus is much larger, the total volatility is much smaller and the investment strategy is more aggressive than that without reinsurance. The main reasons may be that it decreases minimum guaranteed return payment to the consumer due to the decrease of retention and reinsurance helps to decrease underwriting risk[4] and further hedge investment risk so as to obtain more surplus. Therefore, reinsurance not only helps decrease underwriting risks but also increase the surplus of the insurer. This result is conformed to the results obtained by Scordis and Steinorth(2012), in which they find that a positive relation between the use of reinsurance and value.

2.3 Numerical analysis of dynamic programming under the condition without reinsurance

For multi-stage dynamic programming without reinsurance, the results are listed in

4) In our model, we did not consider the volatility of mortality and the underwriting risk is indicated by paying the minimum guaranteed return. When the return rate is lower than the minimum guaranteed return rate, the insurer will have deficit due to resulting overpayment.

Table 4 and Table 6. We change the maturity time from 10 years into 5 years in order to simplify the calculation. Here we assume that the constraint of average annual surplus is $c=0.00$.

<Table 4> The optimal solutions obtained multi−stage dynamic programming under the condition without reinsurance

Stage(k)	1	2	3	4	5
c	0.00				
$g(k)$	0.0048	0.0048	0.0048	0.0093	0.0232
$\alpha_1(k)$	0.00294	0.0294	0.0294	0.0182	0.0138
$\alpha_2(k)$	0.0199	0.0199	0.0199	0.0269	0.0210
$\alpha_3(k)$	0.9507	0.9507	0.9507	0.9549	0.9652
$\min\sigma_k$	0.0051	0.0041	0.0035	0.0027	0.0019
$\frac{E(X(k))}{k}$	0.00002	0.1883	0.1015	0.0572	0.0388

<Table 5> The sub−optimal solutions obtained multi−stage dynamic programming under the condition without reinsurance

Stage(k)	1	2	3	4	5
c	0.00				
$g(k)$	0.0328	0.0328	0.0613	0.0613	0.0699
$\alpha_1(k)$	0.0097	0.0097	0.0256	0.0256	0.0342
$\alpha_2(k)$	0.0749	0.0749	0.0495	0.0495	0.0063
$\alpha_3(k)$	0.9154	0.9154	0.9249	0.9249	0.9674
$\min\sigma_k$	0.0078	0.0065	0.0054	0.0046	0.0028
$\frac{E(X(k))}{k}$	0.0123	0.1877	0.0995	0.0553	0.0342

From Table 4 we find that the optimal solutions of minimum guaranteed return rate are very small except in the last stage. From Table 5 we find that the sub−optimal minimum guaranteed return rate increases with the time, the annual average surplus is small in first year and reaches largest in the second year and then gradually decreases with time from the third year. The total volatility gradually decreases with the time from first year to last year.

2.4. Numerical analysis of dynamic programming under consideration of reinsurance

For multi−stage dynamic programming under consideration of reinsurance, the results are listed in Table 6 and Table 7. From Table 6 we find that the optimal minimum guaranteed return rate are very low except the third stage. From Table 8 we find that the sub−optimal minimum guaranteed return rate, the proportion of retention and investment strategy is same in each stage except those in the first stage. We also find from Table 7 that the sub−optimal minimum guarantee rate is higher and the investment tends to be more aggressive(the portion of risk−free investment is lower) while the sub−optimal retention rate is increasing. The possible explanation may be that higher retention rate means the underwriting risk is not high and the insurer is expecting to obtain more underwriting profit and more investment return, therefore, the policyholder is also expected to obtain more return due to higher minimum guarantee return rate. The optimal total volatilities are larger in every stage than those without considering reinsurance.

<Table 6> The optimal solutions obtained multi−stage dynamic programming under the condition with reinsurance

Stage(k)	1	2	3	4	5
c			0.00		
$Re(k)$	0.5035	0.1838	0.4654	0.5648	0.5035
$g(k)$	0.0093	0.0004	0.0249	0.0057	0.0093
$\alpha_1(k)$	0.0182	0.0597	0.0181	0.0392	0.0182
$\alpha_2(k)$	0.0269	0.0149	0.0191	0.0370	0.0269
$\alpha_3(k)$	0.9549	0.9254	0.9627	0.9238	0.9549

$\min\sigma_k$	0.0044	0.0062	0.0026	0.0047	0.0021
$\frac{E(X(k))}{k}$	0.0242	0.2172	0.1196	0.0649	0.0032

<Table 7> The sub-optimal solutions obtained multi-stage dynamic programming under the condition with reinsurance

Stage (k)	1	2	3	4	5
c	0.00				
$Re(k)$	0.4100	0.7989	0.7989	0.7989	0.7989
$g(k)$	0.0545	0.0638	0.0638	0.0638	0.0638
$\alpha_1(k)$	0.0290	0.0103	0.0103	0.0103	0.0103
$\alpha_2(k)$	0.0607	0.1186	0.1186	0.1186	0.1186
$\alpha_3(k)$	0.9103	0.8712	0.8712	0.8712	0.8712
$\min\sigma_k$	0.0087	0.0109	0.0083	0.0068	0.0058
$\frac{E(X(k))}{k}$	0.0382	0.1670	0.0952	0.0439	0.0382

This result is just opposite to that discussed in Section 2.2. The main reasons may be due to that the proportion of retention is much higher and investment strategy is more aggressive in dynamic programming case. However, the optimal guaranteed return rates are higher except that in last stage. Therefore, it is beneficial to the customer for the universal life insurance with reinsurance as whole.

On the whole, it is important to notice that the investment strategies suggested in our analyses are rather conservative compared with those in current situations(NAIC capital markets special report as of year-end 2010 stated that the portion of bonds and common stock in insurance firms investments in the U.S. were 69.7% and 10.3% respectively, where the bonds includes categories such as corporate dept, municipal bonds, structured securities, U.S. government bonds and foreign government bonds). Since current interest rates are unprecedentedly low in relation to human history, life insurance companies face considerable interest rate risk given their investment in fixed-income securities and their unique liabilities if the interest rate is expected to rise. Moreover, our analysis shows that the optimal minimum guaranteed return rates are much smaller than those in current industry policy(Please see the data in Table 8).

Therefore, we believe that lower portion of bonds and lower guaranteed return rate suggested in our paper can help insurance companies reduce the insolvency risk and avoid the vulnerability to a sustained low interest environment.

<Table 8> Guaranteed interest rate in life insurance policies in the United States

Total Guaranteed Interest Rate				
2006	2007	2008	2009	2010
4.22%	4.20%	4.11%	4.14%	4.09%

Source: National Association of Insurance Commissioners(NAIC), online data at http://www.naic.org/cipr_newsletter_archive/vol3_low_interest_rates.htm

3. Sensitivity analysis of risk-free interest rate and other parameters

3.1 What happens when risk-free interest rate and other parameters change

In the previous sections, it was assumed that the risk-free interest rate and other parameters are unchanged when the valuation models are discussed. In this section, we will discuss the sensitivity of optimal minimum guaranteed return rate, the investment and reinsurance strategy to the change of risk-free interest rate and other parameters. We set the levels of risk-free interest rate, mortality, the insured age and the cost rate of reinsurance 20% higher and lower than the standard values to see what happens when these parameters change(see Table 9).

<Table 9> The optimal and sub-optimal solutions when risk-free interest rate, mortality, insured age and the cost rate of reinsurance changes

$r_f=0.054(1+0.2)$							$\Delta r_f=0.2$	
α_1*	α_2*	α_3*	g*	Re*	$\min\sqrt{\sum_{t=1}^{10}\sigma^2(t)}$	$E^*(X(10)-X(0))$	$\frac{\Delta\sigma}{\sigma(r_f=0.054)}$	$\frac{\Delta E^*(X(10)-X(0))}{E^*(X(10)-X(0))/r_f=0.054}$
0.0996	0.0153	0.8881	0.0901	0.2637	0.000008	0.5784	−0.1111	0.4194
0.1138	0.0019	0.8843	0.0169	0.2525	0.000006	0.7272		
$r_f=0.054(1-0.2)$							$\Delta r_f=-0.2$	
α_1*	α_2*	α_3*	g*	Re*	$\min\sqrt{\sum_{t=1}^{10}\sigma^2(t)}$	$E^*(X(10)-X(0))$	$\frac{\Delta\sigma}{\sigma(r_f=0.054)}$	$\frac{\Delta E^*(X(10)-X(0))}{E^*(X(10)-X(0))/r_f=0.054}$
0.0966	0.0153	0.8881	0.0901	0.2637	0.000010	0.2500	0.1111	−0.3865
0.1138	0.0019	0.8843	0.0169	0.2525	0.000006	0.6010		
$q_{x+t}(1+0.2)$							$\Delta q_{t+x}=0.2$	
α_1*	α_2*	α_3*	g*	Re*	$\min\sqrt{\sum_{t=1}^{10}\sigma^2(t)}$	$E^*(X(10)-X(0))$	$\frac{\Delta\sigma}{\sigma(r_f=0.054)}$	$\frac{\Delta E^*(X(10)-X(0))}{E^*(X(10)-X(0))/r_f=0.054}$
0.0966	0.0153	0.8881	0.0901	0.2637	0.000010	0.3256	0.1111	−0.2010
0.0182	0.0269	0.9549	0.0093	0.5035	0.000003	0.3666		
$q_{x+t}(1-0.2)$							$\Delta q_{t+x}=-0.2$	
α_1*	α_2*	α_3*	g*	Re*	$\min\sqrt{\sum_{t=1}^{10}\sigma^2(t)}$	$E^*(X(10)-X(0))$	$\frac{\Delta\sigma}{\sigma(r_f=0.054)}$	$\frac{\Delta E^*(X(10)-X(0))}{E^*(X(10)-X(0))/r_f=0.054}$
0.029	0.0607	0.9103	0.0545	0.41	0.000005	0.4216	−0.4444	0.0351
0.1138	0.0019	0.8843	0.0169	0.2525	0.000005	0.6483		
$x=40(1+0.2)=48$							$\Delta x=0.2$	
α_1*	α_2*	α_3*	g*	Re*	$\min\sqrt{\sum_{t=1}^{10}\sigma^2(t)}$	$E^*(X(10)-X(0))$	$\frac{\Delta\sigma}{\sigma(r_f=0.054)}$	$\frac{\Delta E^*(X(10)-X(0))}{E^*(X(10)-X(0))/r_f=0.054}$
0.0966	0.0153	0.8881	0.0901	0.2637	0.000009	0.4106	0	0.0076
0.1138	0.0019	0.8843	0.0169	0.2525	0.000005	0.6032		
$x=40(1-0.2)=32$							$\Delta x=-0.2$	
α_1*	α_2*	α_3*	g*	Re*	$\min\sqrt{\sum_{t=1}^{10}\sigma^2(t)}$	$E^*(X(10)-X(0))$	$\frac{\Delta\sigma}{\sigma(r_f=0.054)}$	$\frac{\Delta E^*(X(10)-X(0))}{E^*(X(10)-X(0))/r_f=0.054}$
0.0966	0.0153	0.8881	0.0901	0.2637	0.000009	0.4062	0	−0.0032
0.1138	0.0019	0.8843	0.0169	0.2525	0.000006	0.6010		
$\eta=0.1(1+0.2)=0.12$							$\Delta x=0.2$	
α_1*	α_2*	α_3*	g*	Re*	$\min\sqrt{\sum_{t=1}^{10}\sigma^2(t)}$	$E^*(X(10)-X(0))$	$\frac{\Delta\sigma}{\sigma(r_f=0.054)}$	$\frac{\Delta E^*(X(10)-X(0))}{E^*(X(10)-X(0))/r_f=0.054}$
0.0966	0.0153	0.8881	0.0901	0.2637	0.000009	0.4022	0	−0.0130
0.1138	0.0019	0.8843	0.0169	0.2525	0.000006	0.5966		
$\eta=0.1(1-0.2)=0.08$							$\Delta x=-0.2$	
α_1*	α_2*	α_3*	g*	Re*	$\min\sqrt{\sum_{t=1}^{10}\sigma^2(t)}$	$E^*(X(10)-X(0))$	$\frac{\Delta\sigma}{\sigma(r_f=0.054)}$	$\frac{\Delta E^*(X(10)-X(0))}{E^*(X(10)-X(0))/r_f=0.054}$
0.0966	0.0153	0.8881	0.0901	0.2637	0.000009	0.4128	0	0.0130
0.1138	0.0019	0.8843	0.0169	0.2525	0.000006	0.6067		

From the results of sub−optimum in Table 9, we find that increasing risk−free interest rate, when risk−free interest rate is higher, will increase the optimal surplus of insurance companies and the minimum volatilities of investment and minimum guaranteed return rate, and vise verse. However, the optimal investment, reinsurance strategies and optimal minimum guaranteed return rate keeps same whatever the risk−free interest rate increases or decreases. Table 9 also shows that increasing the mortality will increase the minimum total volatility of investment and guaranteed return rate and decrease the surplus of insurance companies. And the optimal investment, reinsurance strategies and optimal minimum guaranteed return rate keeps same as those when mortality does not change. However, the retention increase, the investment strategy becomes more conservative, minimum guaranteed return rate becomes smaller, the optimal surplus of insurance companies become slightly larger and the minimum total volatility of investment and minimum guaranteed return rate smaller when the mortality decreases. Finally, Table 9 shows that there is little effect of changing the age of the insured and the cost rate of reinsurance on the optimal investment and reinsurance strategies, on the minimum total volatility, and on the minimum guaranteed return rate.

3.2. Discussion

From the results of Table 9, we find that risk−free interest rate is the factor to which response variables are most sensitive(i.e., changes of minimum total volatility and expected surplus of the insurer are the largest in response to this factor). In this section, we will discuss how the optimal solutions change when we take the actual risk−free interest rate in current three months U.S. Federal Government Treasury Bill[5], that is, we take $r_f = 0.1\%$ and observe what happens to the optimal solutions.

5) http://www.marketwatch.com/tools/pftools

Table 10 and Table 11 list the results.

<Table 10> The optimal and sub−optimal solutions of minimum guaranteed return rate and the proportions of investments without consideration of reinsurance ($r_f = 0.1\%$)

C	$\alpha_1 *$	$\alpha_2 *$	$\alpha_3 *$	$g*$	$\min \sqrt{\sum_{t=1}^{10} \sigma^2(t)}$	$E(X(10)-X(0))$
≥0	0.1138	0.0019	0.8843	0.0169	0.0024	0.0290
≥0	0.0294	0.0199	0.9507	0.0048	0.0017	0.0359

<Table 11> The optimal and sub−optimal solutions of minimum guaranteed return rate and the proportions of investments under consideration of reinsurance ($r_f = 0.1\%$)

C	$\alpha_1 *$	$\alpha_2 *$	$\alpha_3 *$	$g*$	Re*	$\min \sqrt{\sum_{t=1}^{10} \sigma^2(t)}$	$E(X(10)-X(0))$
≥0	0.0627	0.0789	0.8583	0.0208	0.2552	0.000010	0.0466
≥0	0.1138	0.0019	0.8843	0.0169	0.2525	0.000006	0.1030

From the result of Table 10 and Table 11 we see that the optimal and sub−optimal solutions of minimum guaranteed return rate are much smaller when risk−free interest rate is very low. However, it is still necessary for insurance companies to set a non−zero but lower level of guaranteed return rate, which will be beneficial for both consumer and the insurer.

II. Conclusions

In this article, we discuss the optimal determination of minimum guaranteed return rate, investment and reinsurance strategies with the help of dynamic programming, stochastic optimization and Monte Carlo techniques. We establish the objective function of minimizing the sum of volatilities of investment and minimum guaranteed return rate, at the same time, satisfying with the constraint of surplus of the insurer larger than a constant. The results shows that changing the insured age, the risk−free interest rate(when risk−free interest rate takes high value), the mortality will not affect the optimal value of minimum guaranteed return rate, investment and reinsurance strategies except the case when the mortality decreases. However, changing these parameters will affect the sum of the volatilities of investment and minimum guaranteed return rate and the surplus of the insurer. The results also show that when risk−free interest rate is very low ($r_f = 0.1\%$), the optimal and sub−optimal minimum guarantee return rates are very low. One major limitation of our study may be that we did not consider the effect of capital, and capital cost, on the optimal strategies and these topics could be studied in future research work.

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요 약

본 논문은 유니버셜 사망보험의 최적 최저보증 수익률과 최적 투자전략 및 재보험전략을 결정하는 방식을 모델링하고 시사점을 제시한다. 이를 위해 잉여금이 사전에 제시된 값에 비해 커야 한다는 제약하에 투자위험과 보증수익 달성 위험을 모두 최소화하는 과정을 거쳤다. 저자는 이러한 최적화 문제의 동태적 해를 구하기 위하여 동적 프로그래밍 방법론을 적용하였다. 또한 무위험 이자율, 피보험자의 나이, 재보험 비용 및 사망률의 변화가 최적 해에 미치는 영향을 살펴보았다. 분석결과 사망률이 감소하는 경우를 제외하고 피보험자의 나이, 무위험 이자율(무위험 이자율이 높은 경우) 및 사망률의 변화는 최적 최저보증 수익률과 최적 투자 및 재보험 전략에 큰 영향을 미치지 않는 것으로 나타났다. 그러나 이러한 변화들은 투자 변동성 및 최저보증 수익률의 변동성의 합과 보험회사의 잉여금에 영향을 미칠 것이다. 연구결과 무위험 이자율이 매우 낮은 상황($r_f = 0.1\%$)에서는 최적 및 준최적 최저보증 수익률도 매우 낮아지는 것으로 나타났다.

※ 국문 색인어: 생명보험, 유니버셜보험, 최저이자율보증, 최적 리스크관리